Tight Bounds for the Price of Anarchy of Simultaneous First Price Auctions

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Abstract

We study the Price of Anarchy of simultaneous first-price auctions for buyers with submodular and subadditive valuations. The current best upper bounds for the Bayesian Price of Anarchy of these auctions are $e/(e-1)$ [34] and 2 [16], respectively. We provide matching lower bounds for both cases *even* for the case of full information and for mixed Nash equilibria via an explicit construction.

We present an alternative proof of the upper bound of $e/(e-1)$ for first-price auctions with fractionally subadditive valuations which reveals the worst-case price distribution, that is used as a building block for the matching lower bound construction.

We generalize our results to a general class of item bidding auctions that we call bid-dependent auctions (including first-price auctions and all-pay auctions) where the winner is always the highest bidder and each bidder's payment depends only on his own bid.

Finally, we apply our techniques to discriminatory price multi-unit auctions. We complement the results of [12] for the case of subadditive valuations, by providing a matching lower bound of 2. For the case of submodular valuations, we provide a lower bound of 1.109. For the same class of valuations, we were able to reproduce the upper bound of $e/(e-1)$ using our non-smooth approach.

1 Introduction

Combinatorial auctions constitute a fundamental, well-studied resource allocation problem that involves the interaction of n selfish agents in competition for m indivisible resources/goods. The preferences of each player for different bundles of the items are expressed via a valuation set function (one per player). The main challenge is to design a *(truthful) mechanism* that allocates the items in an *efficient* way in the equilibrium, i.e., so that it maximizes the *social welfare*, which is the sum of the valuations of the players for the received bundles. Although it is well-known that this can be achieved optimally by the VCG mechanism [36, 11, 18], unfortunately this might take exponential time in m and n [28, 29] (unless P=NP).

In practice, several simple *non-truthful* mechanisms are used. The most notable examples are *generalized second price (GSP)* auctions used by AdWords [13, 35], simultaneous ascending price auctions for wireless spectrum allocation [27], or independent second price auctions on eBay. Furthermore, in these auctions the expressive power of the buyers is heavily restricted by the bidding

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language, so that they are not able to represent their complex preferences precisely. In light of the above, Christodoulou, Kovács and Schapira [9] proposed the study of simple, non-truthful auctions using the *price of anarchy (PoA)* [21] as a measure of inefficiency of such auctions.[1]

Item bidding Of particular interest are the so-called *combinatorial auctions with item-bidding*, from both practical and theoretical aspects. In such an auction, the auctioneer sells each item by running *simultaneously* m independent single-item auctions. Depending on the type of single-item auctions used, the two main variants that have been studied are *simultaneous second-price auctions (SPAs)* [9, 2, 16] and *simultaneous first-price auctions (FPAs)* [20, 34, 16]. In both cases, the bidders are asked to submit a bid for each item. Then each item is assigned to the highest bidder. The main difference is that in the former a winner is charged an amount equal to the second highest bid while in the latter a winner pays his own bid.

FPAs have been shown to be more efficient than SPAs. For general valuations, Hassidim et al. [20] showed that pure equilibria of FPAs are efficient whenever they exist, but mixed, and Bayesian Nash equilibria of FPAs can be highly inefficient in settings with complementarities. For two important classes of valuation functions, namely *fractionally subadditive* and *subadditive*[2], for mixed and Bayesian Nash equilibria, [20, 34] and [16] showed that FPAs have lower (constant) price of anarchy than the respective bounds obtained for SPAs [9, 2, 16]. The current best upper bounds for the price of anarchy in FPA are $e/(e-1)$ for XOS valuations [34], and 2 for subadditive valuations [16] (proven by different techniques).

Our Contribution Following the work of [20, 16, 34], we study the price of anarchy of FPAs for games with complete and incomplete information. Our main concern is the development of tools that provide *tight* bounds for the price of anarchy of these auctions. Our results complement the current knowledge about simultaneous first-price auctions. We provide *matching* lower bounds to the upper bounds by Syrgkanis and Tardos [34] and by Feldman et al. [16], showing that *even* for the case of full information and mixed Nash equilibria the PoA is at least $e/(e-1)$ for submodular[3] valuations (and therefore for XOS) and 2 for subadditive valuations[4].

We present an alternative proof of the upper bound of $e/(e-1)$ for FPAs with *fractionally subadditive* valuations. This bound was shown before in [34] using a general smoothness framework. Our approach does not adhere to their framework. A nice thing with our approach, is that it reveals the worst-case price distribution, that we then use as a building block for the matching lower bound construction. An immediate consequence of our results is that the price of anarchy of these auctions *stays the same*, for mixed, correlated, coarse-correlated, and Bayesian Nash equilibria. Only for pure Nash equilibria it is equal to 1. Our findings suggest that smoothness may provide tight results for certain classes of auctions, using as *a base class* the class of mixed Nash equilibria, and not that of pure equilibria. This is in contrast to what is known for routing games, where the respective base class was the class of pure equilibria.

[1] In this setting, the price of anarchy is defined as the worst-case ratio of the optimal social welfare over the social welfare obtained in a (Bayesian) Nash equilibrium.

[2] Fractionally subadditive valuations are also known as XOS valuations. For definitions of these valuation functions we refer the reader to Section 2.

[3] In fact our lower bound holds even for the class of OXS valuations that is a strict subclass of submodular valuations. We refer the reader to Section 2 for a definition of OXS valuations and for their relation to other valuation classes.

[4] Independently, and after a preliminary version of our work [10], Roughgarden [33] showed a general method to provide lower bounds for the Price of Anarchy of auctions. We discuss it and compare it to our work in the Related Work paragraph.

For buyers with *additive* valuations (or for the single item auction), we show that any mixed Nash equilibrium is *efficient* in contrast to Bayesian Nash equilibria that were previously known not to be always efficient [22]. This suggests an interesting separation between the full and the incomplete information cases as opposed to other valuation functions (for example submodular and subadditive) and other auction formats such as all-pay auctions due to Baye et al. [1].

Then we generalize our results to a class of item bidding auctions that we call *bid-dependent* auctions. Intuitively, a single item auction is bid-dependent if the winner is always the highest bidder, and a bidder's payment depends only on his own bid. Note that both winner and losers may have to pay. Apart from the FPA (where the losers pay 0), another notable item-bidding auction that falls into this class is the simultaneous all-pay (first-price) auction (APA) [34], where all bidders (even the losers) are charged their bids. For subadditive valuations, we show that the PoA of simultaneous bid-dependent auctions is exactly 2, by showing tight upper and lower bounds. We show that the upper bound technique due to Feldman et al. [16] for FPAs, can be applied to *all* mechanisms of this class. Interestingly, although one might expect that FPAs perform strictly better than APAs, our results suggest that all simultaneous bid-dependent auctions perform equally well. We note that our upper bound for subadditive valuations extends the previously known upper bound of 2 for APAs that was only known for XOS valuations [34] .

Finally, we apply our techniques on *discriminatory price multi-unit auctions* [22]. We complement the results by de Keijzer et al. [12] for the case of subadditive valuations, by providing a matching lower bound of 2, for the standard bidding format. For the case of submodular valuations, we were able to provide a lower bound of 1.109. We were also able to reproduce their upper bound of $e/(e-1)$ for submodular bids, using our non-smooth approach. Note that the previous lower bound for such auctions was 1.0004 [12] for Bayesian Nash equilibria. Both of our lower bounds hold for the case of mixed Nash equilibria.

Related Work A long line of research aims to design simple auctions with good performance guarantee (see e.g. [19, 8]). The (in)efficiency of first-price price auctions has been observed in economics (cf. [22]) starting from the seminal work by Vickrey [36]. Bikhchandani [4] was the first who studied the simultaneous sealed bid auctions in full information settings and observed the inefficiency of their equilibria.

Christodoulou, Kovács and Schapira [9] extended the concept of PoA to the Bayesian setting and applied it to item-bidding auctions. Bikhchandani [4] and then Hassidim et al. [20] showed that in case of general valuations, in FPAs pure Nash equilibria are always efficient (whenever they exist), whereas for SPAs Fu, Kleinberg and Lavi [17] proved that the PoA is at most 2. For Bayesian Nash equilibria, Syrgkanis and Tardos [34] and Feldman et al. [16] showed improved upper bounds on the *Bayesian price of anarchy (BPoA)* for FPAs. Syrgkanis and Tardos came up with a general composability framework of smooth mechanisms, that proved to be quite useful, as it led to upper bounds for several settings, such as first price auctions, all-pay auctions and multi-unit auctions.

Only a few lower-bound results are known for the PoA of simultaneous auctions. For valuations that include *complementarities*, Hassidim et al. [20] presented an example with PoA $= \Omega(\sqrt{m})$ for FPA; as suggested in [16], similar lower bound can be derived for SPAs, as well. Under the non-overbidding assumption, Bhawalkar and Roughgarden [2] gave a lower bound of 2.013 for SPAs with subadditive bidders and $\Omega(n^{1/4})$ for correlated bidders. In [16], similar results are shown under the weak non-overbidding assumption. We summarize the PoA results for FPAs in table 1.

Very recently and independently, Roughgarden [33] presented a very elegant methodology to provide PoA lower bounds via a reduction from communication or computational complexity lower bounds for the underlying optimization problem. One consequence of his reduction is a general

Table 1: Summary of the bounds on the PoA of FPAs.

Valuations	Lower Bound		Upper Bound	
General, Pure	1		1	[4] [20]
General, M-B	$\sqrt{m}$	[20]	m	[20]
SA, M-B	2	[This paper]	2	[16]
XOS, M-B	$e/(e-1)$	[This paper]	$e/(e-1)$	[34]
SM, M-B	$e/(e-1)$	[This paper]	$e/(e-1)$	[34]
OXS, M-B	$e/(e-1)$	[This paper]	$e/(e-1)$	[34]

In the first column, the first argument refers to the valuation class and the second argument to the related equilibrium concept. SA and SM stand for subadditive and submodular valuations, respectively, and where 'M-B' appears the bounds hold for mixed, correlated, coarse correlated or Bayesian Nash equilibria.

lower bound of 2 and $e/(e-1)$ for the PoA of *any* simple auction (including item-bidding auctions) with subadditive and fractionally subadditive bidders, respectively. Therefore, there is an overlap with our results for these two classes of valuations. We show these lower bounds via an explicit construction for FPAs (and also for bid-dependent auctions).

We emphasize that these two approaches are incomparable in the following sense. On the one hand, the results in [33] hold for more general formats of combinatorial auctions than the ones we study here. On the other hand, our $e/(e-1)$ lower bound holds even for more special valuation functions where [33]'s results are either weaker ($2e/(2e-1)$ for submodular valuations) or not applicable. For the case of submodular valuations, Feige and Vondrák [15] showed that a strictly higher than $1-1/e$ amount of the optimum social welfare can be obtained in polynomial communication and for gross substitute valuations (and therefore for its subclass, OXS valuations), Nisan and Segal [30] showed that exact efficiency can be obtained in polynomial communication. These two results show that [33]'s technique does not provide tight lower bounds for the settings studied in this paper. We also note that the PoA lower bound obtained by [33]'s reduction can only be applied to approximate Nash equilibria while our results apply to exact Nash equilibria. Further, our PoA lower bound proof for subadditive valuations uses a simpler construction than the proof in [33] and it holds even for the case of only 2 bidders and identical items (multi-unit auction). Finally, it should be stressed that none of our lower bounds for multi-unit auctions can be derived from [33].

Markakis and Telelis [26] studied uniform price *multi-unit* auctions. De Keijzer et al. [12] bounded the BPoA for several formats of multi-unit auctions with first or second price rules. Auctions employing greedy algorithms were studied by Lucier and Borodin [24]. A number of works [31, 6, 32] studied the PoA of generalized SPAs in the full information and Bayesian settings and even with correlated bidders [25]. Chawla and Hartline [7] proved that for the generalized FPAs with symmetric bidders, the pure Bayesian Nash equilibria are unique and always efficient.

Organization of the paper We introduce the necessary background and notation in Section 2. Then, we present the tight bounds for FPAs with fractionally subadditive, subadditive and additive valuations in Sections 3, 4 and 5, respectively. In Section 6, we show how our results can be generalized to a class of auctions that we call bid-dependent. Finally, we apply our techniques to get bounds on the PoA of discriminatory multi-unit auctions in Section 7.

2 Preliminaries

Simultaneous first-price auctions constitute a simple type of combinatorial auctions. In a combinatorial auction with n *players* (or *bidders*) and m *items*, every player $i \in [n]$ has a valuation for each subset of items, given by a valuation function $v_i : 2^{[m]} \to \mathbb{R}_{\geq 0}$, where $v_i \in V_i$ for some possible set of valuations V_i. A valuation profile for all players is $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \times_i V_i$. The v_i functions are monotone and normalized, that is, $S \subseteq T \Rightarrow v_i(S) \leq v_i(T)$, and $v_i(\emptyset) = 0$. We use the short notation $v_i(j) = v_i(\{j\})$.

In the *Bayesian* setting, the valuation of each player i is drawn from V_i according to some known distribution D_i. We assume that the D_i are independent (and possibly different) over the players. In the *full information* setting the valuation v_i is fixed and known by all other players for all $i \in [n]$. Note that the latter is a special Bayesian combinatorial auction, in which player i has valuation v_i with probability 1.

An *allocation* $\mathbf{X} = (X_1, \ldots, X_n)$ is a partition of the items (allowing empty sets X_i), so that each item is assigned to exactly one player. The most common global objective in combinatorial auctions is to maximize the sum of the valuations of the players for their received sets of items, i.e., to maximize the *social welfare* $SW(\mathbf{X})$ of the allocation, where $SW(\mathbf{X}) = \sum_{i \in [n]} v_i(X_i)$. Therefore, for an *optimal allocation* $\mathbf{O}(\mathbf{v}) = \mathbf{O} = (O_1, \ldots, O_n)$ the value $SW(\mathbf{O})$ is maximum among all possible allocations.

In a *simultaneous (or item bidding)* auction every player $i \in [n]$ submits a non-negative bid b_{ij} for each item $j \in [m]$. The items are then allocated by independent auctions: for each $j \in [m]$, the bidder i with the highest bid b_{ij} receives the item. We consider the case when the payment for each item is the *first price* payment: a player pays his own bid (the highest bid) for every item he receives. Our (upper bound) results hold for arbitrary randomized tie-breaking rules. Note that with such a rule, for any fixed $\mathbf{b} = (b_1, \ldots, b_n)$, the probabilities for the players to get a particular item are fixed.

For a given bid vector b_i, item $j \in [m]$ and a subset of items $S \subseteq [m]$ we use the notation $b_i(S) = \sum_{j \in S} b_{ij}$, and $b_i(j) = b_{ij}$. Assume that the players submitted bids for the items according to $b_i = (b_{i1}, \ldots, b_{im})$ and the simultaneous first-price auction yields the allocation $\mathbf{X}(\mathbf{b})$. For simplicity, we use $v_i(\mathbf{b})$ and $SW(\mathbf{b})$ instead of $v_i(X_i(\mathbf{b}))$ and $SW(\mathbf{X}(\mathbf{b}))$, to express the valuation of player i and the social welfare for the allocation $\mathbf{X}(\mathbf{b})$ if $\mathbf{X}$ is clear from the context. The *utility* u_i of player i is defined as his valuation for the received set, minus his payments: $u_i(\mathbf{b}) = v_i(X_i(\mathbf{b})) - b_i(X_i(\mathbf{b}))$.

2.1 Bidding strategies, Nash equilibria, and the price of anarchy

A *pure (bidding) strategy* b_i for player i is a vector of bids for the m items $b_i = (b_{i1}, \ldots, b_{im})$. As usual, $\mathbf{b}_{-i}$ denotes the strategies of all players except for i. The pure strategy profile of all bidders is then $\mathbf{b} = (b_1, b_2, \ldots, b_n)$.

A *mixed strategy* B_i of player i is a probability distribution over pure strategies. Let $\mathbf{B} = (B_1, \ldots, B_n)$ be a profile of mixed strategies. Given a profile $\mathbf{B}$, we fix the notation for the following *cumulative distribution functions (CDF):* unless defined otherwise, G_{ij} is the CDF of the bid of player i for item j; F_j is the CDF of the highest bid for item j in $\mathbf{b}$, and F_{ij} is the CDF of the highest bid for item j in $\mathbf{b}_{-i}$. Observe that $F_j = \Pi_k G_{kj}$, and $F_{ij} = \Pi_{k \neq i} G_{kj}$. We also use $\varphi_{ij}(x)$ to denote the probability that player i gets item j by bidding x. Then $\varphi_{ij}(x) \leq F_{ij}(x)$ due to a possible tie in x.

We review five standard equilibrium concepts studied in this paper: pure, mixed, correlated, coarse correlated and Bayesian Nash equilibria. The first four of them are for the *full information* setting and the last one is defined in the *Bayesian* setting. Let $\mathbf{v} = (v_1, \ldots, v_n)$ be the players valuation functions. In the Bayesian setting, v_i is drawn from V_i according to some known distribution. Let $\mathbf{B}$ denote a distribution over bidding profiles $\mathbf{b}$ of the players. Then, $\mathbf{B}$ is called a

– *pure Nash equilibrium*, if $\mathbf{B}$ is a pure strategy profile $\mathbf{b}$ and $u_i(\mathbf{b}) \geq u_i(b'_i, \mathbf{b}_{-i})$.
– *mixed Nash equilibrium*, if $\mathbf{B} = \times_i B_i$ and $\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})] \geq \mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(b'_i, \mathbf{b}_{-i})]$.
– *correlated Nash equilibrium*, if $\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})|b_i] \geq \mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(b'_i, \mathbf{b}_{-i})|b_i]$.
– *coarse correlated Nash equilibrium*, if $\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})] \geq \mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(b'_i, \mathbf{b}_{-i})]$.
– *Bayesian Nash equilibrium*, if $B_i(\mathbf{v}) = \times_i B_i(v_i)$ and $\mathbb{E}_{\mathbf{v}_{-i},\mathbf{b}\sim\mathbf{B}(\mathbf{v})}[u_i(\mathbf{b})] \geq \mathbb{E}_{\mathbf{v}_{-i},\mathbf{b}_{-i}\sim\mathbf{B}_{-i}(\mathbf{v}_{-i})}[u_i(b'_i, \mathbf{b}_{-i})]$.

where the given inequalities hold for all players i and (pure) deviating bids b'_i. It is well known that each one of the first four classes is contained in the next class, i.e., pure $\subseteq$ mixed $\subseteq$ correlated $\subseteq$ coarse correlated. If we regard the *full information* setting as a special case of the *Bayesian* setting, we also have pure $\subseteq$ mixed $\subseteq$ Bayesian.

For a given auction and fixed valuations $\mathbf{v}$ of the bidders, let $\mathbf{O}$ be an optimal allocation. Then for this auction (game) the *price of anarchy in pure equilibria* is $\mathrm{PoA} = \max_{\mathbf{b}\,\text{pure Nash}} \frac{SW(\mathbf{O})}{SW(\mathbf{b})}$; Given a class of auctions, the *price of anarchy (PoA)* for this type of auction is the worst case of the above ratio, over all auctions of the class, valuation profiles $\mathbf{v}$ and bidding profile B. For the other four types of equilibria, the price of anarchy can be defined analogously.

For the expected utility of a given bidder i we often use the short notation $\mathbb{E}[u_i]$ (if $\mathbf{B}$ is clear from the context) or $u_i(\mathbf{B})$ to denote $\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})]$. Similarly, for fixed b'_i, we use $\mathbb{E}[u_i(b'_i)] = \mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(b'_i, \mathbf{b}_{-i})]$. We also use $\mathbb{E}_{\mathbf{v}}$ instead of $\mathbb{E}_{\mathbf{v}\sim\mathbf{D}}$.

2.2 Types of valuations

Our results concern different classes of valuation functions, which we define next, in increasing order of inclusion. Let $v : 2^{[m]} \to \mathbb{R}_{\geq 0}$, be a valuation function. Then v is called

– *additive*, if $v(S) = \sum_{j\in S} v(j)$;
– *multi-unit-demand* or *OXS*, if for some k there exist k unit demand valuations v^r, $r \in [k]$ (defined as $v^r(S) = \max_{j\in S} v^r(j)$), such that $v(S) = \max_{S=\dot{\bigcup}_{r\in[k]} S_r} \sum_{r\in[k]} v^r(S_r)$;[5]
– *submodular*, if $v(S \cup T) + v(S \cap T) \leq v(S) + v(T)$;
– *fractionally subadditive* or *XOS*, if v is determined by a finite set of *additive* valuations f_γ for $\gamma \in \Gamma$, so that $v(S) = \max_{\gamma\in\Gamma} f_\gamma(S)$;
– *subadditive*, if $v(S \cup T) \leq v(S) + v(T)$;

where the given equalities and inequalities must hold for arbitrary item sets $S, T \subseteq [m]$. It is well-known that each one of the above classes is strictly contained in the next class, e.g., an additive set function is always submodular but not vice versa, a submodular is always XOS, etc. [14]. As an equivalent definition, submodular valuations are exactly the valuations with *decreasing marginal values*, meaning that $v(\{j\} \cup T) - v(T) \leq v(\{j\} \cup S) - v(S)$ holds for any item j, given that $S \subseteq T$.

3 Submodular valuations

In this section we present a lower bound of $\frac{e}{e-1}$ for the mixed PoA in simultaneous first price auctions with OXS and therefore, submodular and fractionally subadditive valuations. This is a matching lower bound to the results by Syrgkanis and Tardos [34].

In order to the explain the key properties of the instance proving a tight lower bound, first we discuss a new approach to obtain the same upper bound for the PoA of a first price single item auction as in [34]. While the upper bound that we derive with the help of this idea, can also be

[5] $\dot{\bigcup}$ stands for disjoint union

obtained based on the very general *smoothness* framework [32, 34, 12], the approach we introduce here does not adhere to this framework.[6] The strength of our approach consists in its potential to lead to better (in this case tight) *lower* bounds, as we demonstrate subsequently.

3.1 PoA Upper Bound for Single Item Auctions

Theorem 3.1. *The PoA of mixed Nash equilibria in first-price single-item auctions is at most* $\frac{e}{e-1}$.

Proof. Let $\mathbf{v} = (v_1, \ldots, v_n)$ be the valuations of the players, and suppose that $v_i = \max_{k\in[n]} v_k$. We fix a mixed Nash equilibrium $\mathbf{B} = (B_1, B_2, \ldots, B_n)$. Let p_i denote the highest bid in $\mathbf{b}_{-i}$, and $F(x) = F_i(x)$ be the cumulative distribution function (CDF) of p_i, that is, $F(x) = \mathbb{P}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[p_i \leq x]$. The following lemma prepares the ground for the selection of an appropriate deviating bid.

Lemma 3.2. *For any pure strategy a of player i, $\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})] \geq F(a)(v_i - a)$.*

Proof. If F is continuous in a, then $F(a) = \mathbb{P}[p_i \leq a] = \mathbb{P}[p_i < a]$, tie-breaking in a does not matter, and $F(a)$ equals also the probability that bidder i gets the item if he bids a. Therefore, $F(a)(v_i - a) = \mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(a, \mathbf{b}_{-i})] \leq \mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})]$, since $\mathbf{B}$ is a Nash equilibrium. If F is not continuous in a ($\mathbb{P}[p_i = a] > 0$), then, as a CDF, it is at least right-continuous. By the previous argument $\mathbb{E}[u_i(\mathbf{b})] \geq F(a+\epsilon)(v_i - a - \epsilon)$ holds for every $x = a + \epsilon$ where F is continuous, and the lemma follows by taking $\epsilon \to 0$. □

Since in a Nash equilibrium the expected utility of every (other) player is non-negative, by summing over all players, it holds that $\sum_{k=1}^{n} \mathbb{E}[u_k(\mathbf{b})] \geq F(a)(v_i - a)$.

On the other hand, for any fixed bidding profile $\mathbf{b}$ we have $u_k(\mathbf{b}) = v_k(X_k(\mathbf{b})) - b_k(X_k(\mathbf{b}))$, where $b_k(X_k(\mathbf{b})) = b_k$ whenever b_k is a winning bid, and $b_k(X_k(\mathbf{b})) = 0$ otherwise. Let $b_{\max}$ be the maximum bid in $\mathbf{b}$. By taking expectations with regard to $\mathbf{b} \sim \mathbf{B}$, and summing over the players, $\mathbb{E}[\sum_k u_k(\mathbf{b})] = \mathbb{E}[\sum_k (v_k(X_k(\mathbf{b})) - b_k(X_k(\mathbf{b})))] = \mathbb{E}[\sum_k v_k(X_k(\mathbf{b})) - b_{\max}] = \mathbb{E}[SW(\mathbf{b})] - \mathbb{E}[b_{\max}]$. By combining this with Lemma 3.2, we obtain

$$\mathbb{E}[SW(\mathbf{b})] = \mathbb{E}\left[\sum_{k=1}^{n} v_k(X_k(\mathbf{b}))\right] \geq F(a)(v_i - a) + \mathbb{E}[b_{\max}] \geq F(a)(v_i - a) + \mathbb{E}[p_i], \tag{1}$$

for any (deviating) bid a. (Analogues of this derivation are standard in the simultaneous auctions literature.) We choose the bid a^* that *maximizes* the right hand side of (1), i.e. $a^* = \arg\max_a F(a)(v - a)$ (see Figure 1 (a) for an illustration). Then, in order to upper bound the PoA, we look for the *maximum* value of λ, such that,

$$F(a^*)(v_i - a^*) + \mathbb{E}[p_i] \geq \lambda v_i. \tag{2}$$

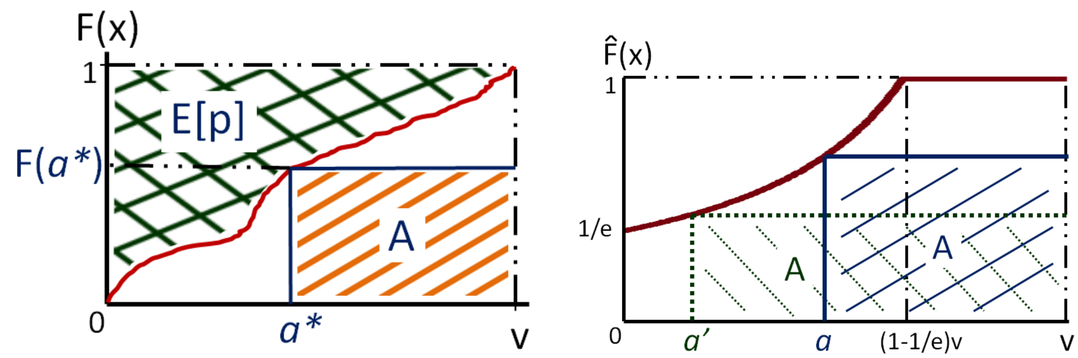
The following lemma settles the maximum value of such λ as $1 - \frac{1}{e}$ for mixed equilibria.[7] This will complete the proof of the theorem, since by (1) and $SW(\mathbf{O}) = v_i$ we obtain $\mathbb{E}[SW(\mathbf{b})] \geq (1 - \frac{1}{e}) \cdot SW(\mathbf{O})$.

Lemma 3.3. *For any non-negative random variable p with CDF F, and any fixed value v, it is true that*

$$F(a^*)(v - a^*) + \mathbb{E}[p] \geq \left(1 - \frac{1}{e}\right) v.$$

[6] Roughly, because the pure deviating bid a that we identify, depends on the other players' bids $\mathbf{b}_{-i}$ in the Nash equilibrium.

[7] If $\mathbf{B}$ is a pure equilibrium, then it is easy to verify that F is a step function, furthermore $a^* = p_i$, and inequality (2) boils down to $1 \cdot (v_i - a^*) + a^* = 1 \cdot v_i$



(a) (b)

Figure 1: Figure (a) is a schematic illustration of the expression $F(a^*)(v-a^*) + \mathbb{E}[p]$, where $A = F(a^*)(v-a^*)$. Figure (b) shows the CDF $\hat{F}(x)$,which makes all the inequalities of Lemma 3.3 tight, i.e. for every $x \in [0, \left(1-\frac{1}{e}\right)v]$, $F(x)(v-x) = A = \frac{v}{e}$.

Proof. Set $A = F(a^*)(v-a^*)$, for $a^* = \arg\max_a F(a)(v-a)$. We use the fact that the expectation of a non-negative random variable with CDF F can be calculated as $\mathbb{E}[x] = \int_0^\infty (1-F(x))dx$. Thus,

$$\begin{aligned} F(a^*)(v-a^*) + \mathbb{E}[p] &\geq A + \int_0^{v-A} (1-F(x))dx = A + (v-A) - \int_0^{v-A} F(x)dx \\ &\geq v - \int_0^{v-A} \frac{A}{v-x}dx = v + A\ln\left(\frac{A}{v}\right) \\ &\geq v + \frac{v}{e}\ln\left(\frac{1}{e}\right) = \left(1-\frac{1}{e}\right)v, \end{aligned}$$

where the last inequality is due to the fact that $A\ln(\frac{A}{v})$ is minimized for $A = \frac{v}{e}$. ☐

☐

Worst-case price distribution The CDF $F(x)$ that makes all the inequalities of (the proof of) Lemma 3.3 tight (see Figure 1(b)), is

$$\hat{F}(x) = \begin{cases} \frac{v}{e(v-x)} & \text{, for } x \leq \left(1-\frac{1}{e}\right)v \\ 1 & \text{, for } x > \left(1-\frac{1}{e}\right)v \end{cases}$$

Observe that for $x \leq \left(1-\frac{1}{e}\right)v$, $\hat{F}(x)(v-x) = \frac{v}{e}$ and for $x > \left(1-\frac{1}{e}\right)v$, $\hat{F}(x)(v-x) = v-x < v-(1-\frac{1}{e})v = \frac{v}{e}$. So, the bid that maximizes the quantity $\hat{F}(a)(v-a)$ is any value $a \in [0, \left(1-\frac{1}{e}\right)v]$. The given distribution $\hat{F}$ for p_i makes inequality (2) tight. In order to construct a (tight) lower bound for the PoA, we also need to tighten the inequalities in (1). Note that the inequality of Lemma 3.2 is tight for *all* $a \in [0, \left(1-\frac{1}{e}\right)v]$. Intuitively, we need to construct a Nash equilibrium, where the CDF of p_i is equal to $\hat{F}(x)$ and b_i doesn't exceed p_i. We present a construction (with many items) in Section 3.2.

Remark 1. Here we discuss our technique and the smoothness technique that achieves the same upper bound [34]. In [34], a particular mixed bidding strategy A_0 was defined for each player i, such that for every price $p = \max_{i' \neq i} b_{i'}$, $\mathbb{E}_{A_0}[u_i(A_0, p)] + p \geq v(1-1/e)$. If we denote $g(A, F) = \mathbb{E}_{A,F}[u_i(a,p)+p]$, it can be deduced that $\max_A \min_p g(A,p) \geq v(1-1/e)$. In Lemma 3.3 we show that $\min_F \max_a g(a, F) \geq v(1-1/e)$. Moreover, we prove that the inequality is tight by providing the *minimizing* distribution $\hat{F}$, such that $\max_a g(a, \hat{F}) = v(1-1/e)$. By the Minimax Theorem, $\min_F \max_a g(a, F) = \max_A \min_p g(A, p) = v(1-1/e)$. One advantage of our approach is that it can be coupled with a worst-case distribution $\hat{F}$ that serves as an optimality certificate of the method. Moreover, if one can convert $\hat{F}$ to Nash Equilibrium strategy profile (see Section 3.2), a tight Nash equilibrium construction is obtained; this can be a challenging task, though.

3.2 Tight Lower Bound

Here we present a tight lower bound of $\frac{e}{e-1}$ for the mixed PoA in simultaneous first price auctions with OXS valuations. This implies a lower bound for submodular and fractionally subadditive valuations.

Theorem 3.4. *The price of anarchy of simultaneous first price auctions with full information and OXS valuations is at least* $\frac{e}{e-1} \approx 1.58$.

Proof. We construct an instance with $n+1$ players and n^n items. We define the set of items as $M = [n]^n$, that is, they correspond to all the different vectors $w = (w_1, w_2, ..., w_n)$ with $w_i \in [n]$ (where $[n]$ denotes the set of integers $\{1, \ldots, n\}$). Intuitively, they can be thought of as the nodes of an n dimensional grid, with coordinates in $[n]$ in each dimension.

We call player 0 the *dummy* player, and all other players $i \in [n]$ *real* players. We associate each *real* player i with one of the dimensions (directions) of the grid. In particular, for any fixed player i, his valuation for a subset of items $S \subseteq M$ is the size (number of elements) in the $n-1$-dimensional projection of S in direction i. Formally,

$$v_i(S) = |\{w_{-i} \,|\, \exists w_i \ s.t. \ (w_i, w_{-i}) \in S\}|.$$

It is straightforward to check that v_i has decreasing marginal values, and is therefore submodular[8]. The *dummy* player 0 has valuation 0 for any subset of items.

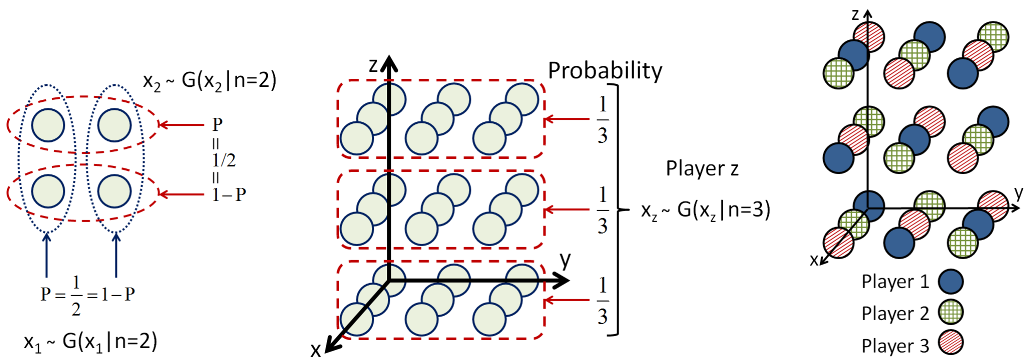
Given these valuations, we describe a mixed Nash equilibrium $\mathbf{B} = (B_1, \ldots, B_n)$ having a PoA arbitrarily close to $e/(e-1)$, for large enough n. The dummy player bids 0 for every item, and receives the item if all of the real players bid 0 for it. The utility and welfare of the dummy player is always 0. For real players the mixed strategy B_i is the following. Every player i picks a number $\ell \in [n]$ uniformly at random, and an x according to the distribution with CDF

$$G(x) = (n-1)\left(\frac{1}{(1-x)^{\frac{1}{n-1}}} - 1\right),$$

where $x \in \left[0, 1 - \left(\frac{n-1}{n}\right)^{n-1}\right]$. Subsequently, he bids x for every item $w = (\ell, w_{-i})$, with $w_i = \ell$ as i^{th} coordinate, and bids 0 for the rest of the items, see Figure 2 for the cases of $n = 2$ and $n = 3$. That is, in any b_i in the support of B_i, the player bids a positive x only for an $n-1$ dimensional slice of the items. Observe that $G(\cdot)$ has no mass points, so tie-breaking matters only in case of 0 bids for an item, in which case player 0 gets the item.

Let $F(x)$ denote the probability that bidder i gets a fixed item j, given that he bids $b_i(j) = x$ for this item, and the bids in $\mathbf{b}_{-i}$ are drawn from $\mathbf{B}_{-i}$ (due to symmetry, this probability is the same for all items $w = (\ell, w_{-i})$). For every other player k, the probability that he bids 0 for item j

[8] These valuations are also OXS. In the definition of OXS valuations (Section 2), we set $k = n^{n-1}$ and for the unit-demand valuations corresponding to player i the following holds: if item j corresponds to $w = (w_1, w_2, ..., w_n)$ then for $r \in [k]$, $v_i^r(j) = 1$, if w_{-i} is the n-ary representation of r and $v_i^r(j) = 0$, otherwise.



(a) (b) (c)

Figure 2: The figure illustrates the cases $n = 2$ and $n = 3$ ((a) and (b) respectively) for the lower bound example with submodular valuation functions. In (c) an optimal allocation for the case $n = 3$ is shown.

is $(n-1)/n$, and the probability that j is in his selected slice but he bids lower than x is $G(x)/n$. Multiplying over the $n-1$ other players, we obtain

$$F(x) = \left(\frac{G(x)}{n} + \frac{n-1}{n}\right)^{n-1} = \frac{\left(\frac{n-1}{n}\right)^{n-1}}{1-x}.$$

Notice that v_i is an additive valuation restricted to the slice of items that player i bids for in a particular b_i. Therefore, when player i bids x in b_i, his expected utility is $F(x)(1-x)$ for one of these items, and comprising all items it is $\mathbb{E}[u_i(b_i)] = n^{n-1}F(x)\cdot(1-x) = n^{n-1}\left(\frac{n-1}{n}\right)^{n-1} = (n-1)^{n-1}$.

Next we show that $\mathbf{B}$ is a Nash equilibrium. In particular, the bids b_i in the support of B_i maximize the expected utility of a fixed player i.

First, we fix an arbitrary w_{-i}, and focus on the set of items $C := \{(\ell, w_{-i}) \mid \ell \in [n]\}$, which we call a *column* for player i. Recall that i is interested in getting only one item within C, while his valuation is additive over items from different columns. Moreover, in a fixed $\mathbf{b}_{-i}$, every other player k submits the same bid for all items in C, because either the whole C is in the current slice of k, and he bids the same value x, or no item from the column is in the slice and he bids 0. Consider first a deviating bid, in which i bids a positive value for more than one items in C, say (at least) the values $x \geq x' > 0$ where x is his highest bid in C. Then his expected utility for this column is strictly less than $F(x)(1-x)$, because his value is $F(x)\cdot 1$, but he might have to pay $x + x'$, in case he gets both items. Consequently, bidding x for only one item in C and 0 for the rest of C is more profitable.

Second, observe that restricted to a fixed column, submitting any bid $x \in \left[0, 1-\left(\frac{n-1}{n}\right)^{n-1}\right]$ for one arbitrary item results in the constant expected utility of $\left(\frac{n-1}{n}\right)^{n-1}$, whereas a bid higher than $1-\left(\frac{n-1}{n}\right)^{n-1}$ guarantees the item but pays more so the utility becomes strictly less than $\left(\frac{n-1}{n}\right)^{n-1}$ for this column. In summary, bidding for exactly one item from each column, an arbitrary (possibly different) bid $x \in \left[0, 1-\left(\frac{n-1}{n}\right)^{n-1}\right]$ is a best response for i yielding the above expected utility, which concludes the proof that $\mathbf{B}$ is a Nash equilibrium.

It remains to calculate the expected social welfare of $\mathbf{B}$, and the optimal social welfare. We define a random variable w.r.t. the distribution $\mathbf{B}$. Let $Z_j = 1$ if one of the real players $1, \ldots, n$ gets item j, and $Z_j = 0$ if player 0 gets the item. Note that the social welfare is the random variable

$\sum_{j\in M} Z_j$, and the expected social welfare is

$$\mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[SW(\mathbf{b})] = \sum_j E[Z_j]\cdot 1 = n^n(1 - Pr(\text{no real player bids for j})) = n^n\left(1-\left(\frac{n-1}{n}\right)^n\right).$$

Finally, we show that the optimum social welfare is n^n. An optimal allocation can be constructed as follows: For each item $(w_1, w_2, ..., w_n)$ compute $r = (\sum_{i=1}^n w_i \mod n)$. Allocate this item to the player $r+1$. It is easy to see that this way the n items of *any* particular column $\{(\ell, w_{-k}) \mid \ell \in [n]\}$ (in any direction k) are given to the n different players, and that each player is allocated n^{n-1} items (Figure 2(c) shows the optimum allocation for $n = 3$). In other words, any two items allocated to the same player differ in at least two coordinates. In particular, they belong to different columns of *this player*, and all contribute 1 to the valuation of the player, which is therefore n^{n-1}. Since this valuation is maximum possible for every player, the obtained social welfare of n^n is optimal.

Thus, the Price of Anarchy is $\frac{1}{(1-(\frac{n-1}{n})^n)}$, and for large n it converges to $\frac{1}{(1-\frac{1}{e})} \approx 1.58$. □

4 Subadditive valuations

Here we show a lower bound of 2 on the mixed PoA when players have subadditive valuations. This lower bound matches the upper bound by Feldman et al. [16].

Theorem 4.1. *The mixed PoA of simultaneous first price auctions with subadditive bidder valuations is at least* 2*.*

Proof. Consider two players and m items with the following valuations: player 1 is a unit-demand player with valuation $v < 1$ (to be determined later) if she gets at least one item; player 2 has valuation 1 for getting at least one but less than m items, and 2 if she gets all the items. Inspired by [20], we use the following distribution functions:

$$G(x) = \frac{(m-1)x}{1-x} \qquad x \in [0, 1/m]; \qquad\qquad F(y) = \frac{v - 1/m}{v-y} \qquad y \in [0, 1/m].$$

Player 1 picks one of the m items uniformly at random, and bids x for this item and 0 for all other items. Player 2 bids y for each of the m items. The bids x and y are drawn from distributions with CDF $F(x)$ and $G(y)$, respectively. In the case of a tie, the item is always allocated to player 2.

Let $\mathbf{B}$ denote this mixed bidding profile. We are going to prove that $\mathbf{B}$ is a mixed Nash equilibrium for every $v > 1/m$.

If player 1 bids any x in the range $(0, 1/m]$ for the one item, she gets the item with probability $F(x)$, since a tie appears with zero probability. Her expected utility for $x \in (0, 1/m]$ is $F(x)(v-x) = v - 1/m$. Thus if player 1 picks x randomly according to $G(x)$, her utility is $v - 1/m$ (note that according to $G(x)$ she bids 0 with zero probability). Bidding something greater than $1/m$ results in a utility less than $v - 1/m$. Regarding player 1, it remains to show that her utility while bidding for only one item is at least her utility while bidding for more items. Suppose player 1 bids x_i for item i, $1 \le i \le m$. W.l.o.g., assume $x_i \ge x_{i+1}$, for $1 \le i \le m-1$. Player 1 doesn't get any item if and only if $y \ge x_1$. So, with probability $F(x_1)$, she gets at least one item and she pays at least x_1. Therefore, her expected utility is at most $F(x_1)(v - x_1) = v - 1/m$, (but it would be strictly less if she is charged nonzero payments for other items). This means that bidding only x_1 for one item and zero for the rest of them dominates the strategy we have assumed.

If player 2 bids a common bid y for all items, where $y \in [0, 1/m]$, she gets m items with probability $G(y)$ and $m-1$ items with probability $1 - G(y)$. Her expected utility is $G(y)(2 - my) +$

$(1-G(y))(1-(m-1)y) = G(y)(1-y)+1-(m-1)y = 1$. We show that player 2 cannot get a utility higher than 1 by using any deviating bids. Suppose now that player 2 bids y_i for item i, for $1 \le i \le m$. Player 1 bids for item i (according to $G(x)$) with probability $1/m$. We also use that since G is a CDF, for $x > 1/m$ holds that $G(x) = 1 < \frac{(m-1)x}{1-x}$. So, the expected utility of player 2 is

$$\begin{aligned}
& \frac{1}{m}\sum_{i=1}^{m}\left(G(y_i)\left(2-\sum_{j=1}^{m} y_j\right) + (1-G(y_i))\left(1-\sum_{\substack{j=1\\ j\neq i}}^{m} y_j\right)\right) \\
= \; & \frac{1}{m}\sum_{i=1}^{m}\left(G(y_i)(1-y_i)+1-\sum_{\substack{j=1\\ j\neq i}}^{m} y_j\right) \\
\le \; & \frac{1}{m}\sum_{i=1}^{m}\left(\frac{(m-1)y_i}{1-y_i}(1-y_i)+1-\sum_{\substack{j=1\\ j\neq i}}^{m} y_j\right) \\
= \; & \frac{1}{m}\sum_{i=1}^{m}\left(my_i+1-\sum_{j=1}^{m} y_j\right) \\
= \; & \frac{1}{m}\left(m\sum_{i=1}^{m} y_i + m - m\sum_{j=1}^{m} y_j\right) = 1.
\end{aligned}$$

Overall, we proved that $\mathbf{B}$ is a mixed Nash equilibrium.

It is easy to see that the optimal allocation gives all items to player 2, and has social welfare 2. In the Nash equilibrium $\mathbf{B}$, player 2 bids 0 with probability $1-\frac{1}{mv}$, so, with at least this probability, player 1 gets one item.

$$SW(\mathbf{B}) \le \left(1-\frac{1}{mv}\right)(v+1) + \frac{1}{mv}2 = 1+v+\frac{1}{mv}-\frac{1}{m}$$

If we set $v = 1/\sqrt{m}$, then $SW(\mathbf{B}) \le 1+\frac{2}{\sqrt{m}}-\frac{1}{m}$. So, PoA$\ge \frac{2}{1+\frac{2}{\sqrt{m}}-\frac{1}{m}}$ which, for large m, converges to 2. $\square$

5 Additive valuations

For additive valuations, we show that mixed Nash equilibria are *efficient*, whenever they exist. This implies an interesting separation between mixed equilibria with full information and Bayesian equilibria, that are known not to be efficient [22]. For the sake of completeness, we present a lower bound of 1.06 for the Bayesian PoA of single-item auctions, in Appendix B.

5.1 The PoA for single item auctions is 1

We consider a first-price single-item auction, where the valuations of the players for the item are given by $(v_1, v_2, \dots, v_n)$. We show that the PoA in mixed strategies is 1. For any mixed

Nash equilibrium of strategies $\mathbf{B} = (B_1, B_2, \ldots, B_n)$, let B_i denote the probability measure of the distribution of bid b_i; in particular, $B_i(I) = \mathbb{P}[b_i \in I]$ for any real interval I. The corresponding cumulative distribution function (CDF) of b_i is denoted by G_i (i.e., $G_i(x) = B_i((-\infty, x])$). Recall that for a given $\mathbf{B}$, for every bidder $i \in [n]$, $F_i(b_i)$ denotes the CDF of $max_{j \neq i} b_j$. We also use $\varphi_i(b_i)$ to denote the probability that player i gets the item with bid b_i. Note that $\varphi(b_i) \leq F_i(b_i)$, due to a possible tie at b_i. Therefore, if he bids b_i, then his expected utility is

$$\mathbb{E}[u_i(b_i)] = \mathop{\mathbb{E}}_{\mathbf{b}_{-i} \sim \mathbf{B}_{-i}}[u_i(b_i)] = \varphi_i(b_i)(v_i - b_i) \leq F_i(b_i)(v_i - b_i).$$

Let $\mathbb{E}[u_i] = \mathbb{E}_{\mathbf{b} \sim \mathbf{B}}[u_i]$ denote his overall expected utility defined by the (Lebesgue) integral $\mathbb{E}[u_i] = \int_{(-\infty,\infty)} \varphi_i(x)(v_i - x) dB_i$. Furthermore, assuming $\mathbb{P}[b_i \in I] > 0$ for some interval I, let $\mathbb{E}[u_i | b_i \in I] = \mathbb{E}_{\mathbf{b} \sim \mathbf{B}}[u_i | b_i \in I]$ be the expected utility of i, on condition that his bid is in I. By definition

$$\mathbb{E}[u_i | b_i \in I] = \frac{\int_I \varphi_i(x)(v_i - x) dB_i}{\mathbb{P}[b_i \in I]} \leq \frac{\int_I F_i(x)(v_i - x) dB_i}{\mathbb{P}[b_i \in I]}.$$

The next lemma follows from the definition of (mixed) Nash equilibria. It states that in equilibrium the excepted utility of any player i, conditioned on the event $b_i \in I$, must be equal to his overall expected utility, given that he bids with positive probability in the interval I.

Lemma 5.1. *In any mixed Nash equilibrium* $\mathbf{B}$, *for every player* i *holds that if* $\mathbb{P}[b_i \in I] > 0$ *, then* $\mathbb{E}[u_i | b_i \in I] = \mathbb{E}[u_i]$.

Proof. Assume that $\mathbb{E}[u_i | b_i \in I] > \mathbb{E}[u_i]$. Then, the player would be better off by submitting bids only in the interval I (according to the distribution $B'_i(I') = B_i(I')/B_i(I)$ for all $I' \subset I$). If $\mathbb{E}[u_i | b_i \in I] < \mathbb{E}[u_i]$, the proof is analogous: in this case the player would be better off by bidding outside the interval. Both cases would contradict $\mathbf{B}$ being a Nash equilibrium. □

In the next lemma we show that for any two players with positive utility, the infimum of their bids must be equal, and they both bid higher than this value with probability 1. Intuitively, if a player has non-zero utility, then his lowest possible bid cannot be lower than any player's lowest bid.

Lemma 5.2. *Assume that in a mixed Nash equilibrium* $\mathbf{B}$ *there are bidders* i *and* j, *with positive utilities* $\mathbb{E}[u_i] > 0$, *and* $\mathbb{E}[u_j] > 0$. *Let* $q_i = \inf_x\{G_i(x) > 0\}$ *and* $q_j = \inf_x\{G_j(x) > 0\}$. *Then* $q_i = q_j = q$, *and* $G_i(q) = G_j(q) = 0$, *consequently* $F_i(q) = F_j(q) = 0$.

Proof. Assume w.l.o.g. that $q_i > q_j$. Note that, by the definition of q_j, player j bids with positive probability in the interval $I = [q_j, q_i)$.

On the other hand, $F_j(x) = 0$ over interval I, since (at least) player i bids higher than x with probability 1. This implies $\mathbb{E}[u_j | b_j \in I] = 0$. Using Lemma 5.1 we obtain $\mathbb{E}[u_j] = \mathbb{E}[u_j | b_j \in I] = 0$, contradicting our assumptions. This proves $q_i = q_j = q$.

Next we show $G_i(q) = G_j(q) = 0$. Observe first, that because of $\mathbb{E}[u_j] > 0$, $v_j > q$ and $v_i > q$ must hold, since q is the smallest possible bid of j and of i. Assume now that $G_i(q) > 0$ and $G_j(q) = 0$. Then, $\mathbb{P}[b_i = q] > 0$, but $\mathbb{E}[u_i | b_i = q] = 0$, since j bids higher. This contradicts again Lemma 5.1 for the interval $[q, q]$.

Second, assume that $G_i(q) > 0$ and $G_j(q) > 0$. In case $b_i = b_j = q$, bidder i or bidder j receives the item with probability smaller than 1. W.l.o.g., we assume it is player i. In this case bidder i is better off by bidding $q + \epsilon$ for a small enough ϵ instead of bidding q, since in case of bids $b_i = q + \epsilon$, and $b_j = q$, he gets the item for sure. This contradicts $\mathbf{B}$ being a Nash equilibrium, and altogether we conclude $G_i(q) = G_j(q) = 0$.

Finally, this immediately implies $F_i(q) = F_j(q) = 0$, since for both i and j, (at least) the other one bids higher than q with probability 1. $\square$

Finally, we prove that mixed equilibria are always efficient. We use the above lemma to show that all players who have non-zero utility, must have maximum valuation.

Theorem 5.3. *In a single-item auction the PoA of mixed Nash equilibria is 1.*

Proof. Let v_i be the maximum valuation in the single item auction with full-information. Assume for the sake of contradiction that a mixed Nash equilibrium $\mathbf{B}$ has $\mathbb{E}_{\mathbf{b}\in\mathbf{B}}[SW(\mathbf{b})] < SW(OPT) = v_i$. Then, there is a nonempty set of bidders $J \subset [n] \setminus \{i\}$, who all get the item with *positive* probability in $\mathbf{B}$, moreover $v_k < v_i$ holds for all $k \in J$. Let $j \in J$ denote the player with maximum valuation $v_j < v_i$ among players in J.

We show that $\mathbb{E}[u_i] > 0$, and $\mathbb{E}[u_j] > 0$. Let us first consider the distribution $F_i(x)$ of the maximum bid in $\mathbf{b}_{-i}$. If $F_i(v_i - \delta) = 0$ for all $\delta > 0$, then the highest bid in $\mathbf{b}_{-i}$, and thus the payment of player j is at least $v_i > v_j$ whenever j wins the item. Thus for his utility $\mathbb{E}[u_j] < 0$, contradicting that $\mathbf{B}$ is a Nash equilibrium. Therefore, there exists a small δ, such that $F_i(v_i-\delta) > 0$. This implies $\mathbb{E}[u_i] > 0$, because by bidding $v_i - \delta/2$ only, player i would have higher than 0 utility.

Now assume for the sake of contradiction that $\mathbb{E}[u_j] = 0$ ($\mathbb{E}[u_j] < 0$ is impossible in an equilibrium). Note that $F_j(v_j) > 0$, otherwise j would get the item with positive probability, but always for a price higher than v_j. On the other hand, if there were a small δ' such that $F_j(v_j - \delta') > 0$, then j could improve his 0 utility by bidding $v_j - \delta'/2$ only. The latter implies, that j can get the item (with positive probability) only with bids v_j or higher, so he never bids higher so as to avoid negative expected utility. We obtained that $\varphi_j(v_j) > 0$, where $\varphi_j(v_j)$ denotes the probability of j winning the item with bid v_j.

Moreover, $F_j(v_j - \delta') = 0$ for all $\delta' > 0$ implies that the minimum bid of at least one player k is at least v_j ($\inf_x\{G_k(x) > 0\} \geq v_j$). Therefore the *winning* bids of player i are also at least v_j (both when $i = k$, and when $i \neq k$). But then i could improve his utility by overbidding the (with positive probabilty) winning bid v_j of j, that is, by bidding exactly $v_j + \epsilon$ (instead of $\leq v_j + \epsilon$) with probability $G_i(v_j + \epsilon)$ for a small enough ϵ. With this bid, the *additional* utility of i would get arbitrarily close to (at least) $\varphi_j(v_j)(v_i - v_j) > 0$.

Thus, we established the existence of players i and j, with different valuations $v_j < v_i$, and both with strictly positive expected utility in $\mathbf{B}$. According to Lemma 5.2, for the infimum of these two players' bids $q_i = q_j = q$, and $F_i(q) = F_j(q) = 0$ hold. Furthermore, $q < v_j < v_i$, otherwise the utility of j could not be positive. By the definition of $q_i = q$, for any $\epsilon > 0$ it holds that $\mathbb{P}[q \leq b_i < q + \epsilon] > 0$. Therefore, by Lemma 5.1, and by the definition of conditional expectation, for the interval $I = [q, q + \epsilon)$ we have

$$\begin{aligned} \mathbb{E}[u_i] &= \mathbb{E}[u_i | b_i \in I] \leq \frac{\int_I F_i(x)(v_i - x)dB_i}{\mathbb{P}[b_i \in I]} \\ &< \frac{\int_I F_i(q+\epsilon)(v_i - q)dB_i}{\mathbb{P}[b_i \in I]} = F_i(q+\epsilon)(v_i - q). \end{aligned}$$

Rearranging terms, this yields $F_i(q + \epsilon) > \mathbb{E}[u_i]/(v_i - q) > 0$ for arbitrary $\epsilon > 0$. Since $F_i(x)$ as a cumulative distribution function is right-continuous in every point, this positive lower bound must hold for $F_i(q)$ as well, contradicting $F_i(q) = 0$. $\square$

5.2 Upper bound for additive valuations

We extend the above proof for additive valuations.

Theorem 5.4. *For simultaneous first-price auctions with additive valuations the PoA of mixed Nash equilibria is 1.*

Proof. Let $\mathbf{B}$ be a mixed Nash equilibrium in the m item case. We argue first that for any fixed bidder i, it is without loss of generality to assume that in B_i his bids for each item are drawn from independent distributions. If this were not the case, we could determine the distribution B_i^j of $b_i(j)$ for any item to have the same CDF G_i^j as the distribution of bids for this item in B_i. Then we would replace B_i by the product distribution for the items $B_i' = \times B_i^j$. Since both the expected valuation and the expected payment for item j would remain the same in this new strategy, and the valuation and utility of the player are the sum of valuations and utilities over the items, none of these amounts would be affected. Furthermore the same additivity holds for any other player k, whose 'price function' $F_k^j(\cdot)$ for item j would also not be influenced. Thus, with B_i replaced by the strategy B_i', the mixed profile $\mathbf{B}' = (B_i', B_{-i})$ would remain a mixed Nash with the same expected social welfare as $\mathbf{B}$.

The remaining argument is similar. Now the distribution of bids $(B_i^j)_{i\in[n]}$ for any particular item j corresponds to a mixed Nash equilibrium of the single item auction for this item. Otherwise a player could improve his utility for j, and consequently the sum of his utilities for all items. In turn, by Theorem 5.3 this implies that the social welfare for each item j is optimal, a player (or players) with maximum valuation receive the item, which concludes the proof. □

6 Bid-Dependent Auctions

Here we generalize some of our results to simultaneous bid-dependent auctions. Intuitively, a single item auction is *bid-dependent* if the winner is always the highest bidder, and a bidder's payment depends only on whether she gets the item or not, and on her *own* bid. For instance, the first-price auction and the all-pay auction are bid-dependent but the second-price auction is not.

For a given simultaneous bid-dependent auction, we will denote by $q_j^w(x)$ and $q_j^l(x)$ a bidder's payment $p_{ij}(\mathbf{b})$ for item j when her bid for j is x, depending on whether she is the winner or a loser, respectively. Note that we assume $q_j^w(x)$ (resp. $q_j^l(x)$) to be the same for all bidders. Without this assumption the PoA is unbounded, as we show in Appendix C. To guarantee the existence of reasonable Nash Equilibria, we also make the following natural assumptions about $q_j^w(x)$ and $q_j^l(x)$:[9]

– $q_j^w(x)$ and $q_j^l(x)$ are non-decreasing, continuous functions of x and normalized, such that $q_j^l(0) = q_j^w(0) = 0$;

– $q_j^w(x) \geq q_j^l(x)$ for all $x \geq 0$;

– $q_k^w(x) > 0$ for some x (to avoid the case of all payments being zero, for that no Nash equilibria exist).

6.1 Fractionally Subadditive valuations

6.1.1 Upper Bounds

In this section we discuss the general upper bound for simultaneous bid-dependent auctions.

We define θ as $\theta = \max_{j\in[m]} \sup_{\{x:q_j^w(x)\neq 0\}}\{q_j^l(x)/q_j^w(x)\}$.

Observe that $\theta \in [0,1]$, due to the assumption $q_j^l(x) \leq q_j^w(x)$. We will prove that (for $\theta \neq 1$) the coarse-correlated and the Bayesian PoA of simultaneous bid-dependent auctions with fractionally

[9]Similar assumptions are also made in [3], [23] and [5].

subadditive bidders is at most $\frac{(\theta-1)^2}{\theta^2-\theta+1-e^{\theta-1}}$. When we set $\theta = 0$ or $\theta \to 1$, we get back the upper bounds of $e/(e-1)$ for first-price auctions, and 2 for all-pay auctions, respectively.

We start by proving a lemma for a single item, analogous to Lemma 3.3.

Lemma 6.1. *Consider a single item bid-dependent auction with payment functions $q^w(x)$ and $q^l(x)$. Let $\mathbf{B}$ be an arbitrary randomized bidding profile, and F_i denote the CDF of the random variable $\max_{k\neq i} b_k$, for this $\mathbf{B}$. Then for every bidder i, and non-negative value v, there exists a pure bidding strategy $a = a(v, \mathbf{B}_{-i})$ such that,*

$$F_i(a)\left(v - q^w(a) + q^l(a)\right) - q^l(a) + \sum_{k\in[n]} p_k(\mathbf{B}) \geq \frac{\theta^2-\theta+1-e^{\theta-1}}{(\theta-1)^2}\cdot v,$$

where $p_k(\mathbf{B}) = \mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[p_k(\mathbf{b})]$ is the expected payment from player k.

Proof. Let $a = \arg\max_x \left\{F_i(x)\left(v - q^w(x) + q^l(x)\right) - q^l(x)\right\}$ and $A = F_i(a)\left(v - q^w(a) + q^l(a)\right) - q^l(a)$. In the following we use that F_i is the CDF of $\max_{k\neq i} b_k$, and since $q^w(\cdot)$ is continuous, $\mathbb{E}[q^w(\max_{k\neq i} b_k)] = \int_0^\infty (1-F_i(x))dq^w(x)$ holds.

$$\begin{aligned}
A + \sum_k p_k(\mathbf{B}) &\geq A + \mathbb{E}_{\mathbf{b}}[q^w(\max_{k\neq i} b_k)] \\
&= A + \int_0^\infty (1-F_i(x))dq^w(x) \\
&\geq A + \int_0^\infty \left(1 - \frac{A + q^l(x)}{v - q^w(x) + q^l(x)}\right) dq^w(x) \\
&\geq A + \int_0^\infty \left(\frac{v - A - q^w(x)}{v + (\theta-1)q^w(x)}\right) dq^w(x) \\
&\geq A + \int_0^{v-A} \left(\frac{v-A-y}{v+(\theta-1)y}\right) dy
\end{aligned}$$

The second inequality follows from the definition of A and a and the third one is due to the fact that $q_j^l(x) \leq \theta \cdot q_j^w(x)$ for any x. For the last one, $q^w(0) = 0$ and we further need to show that for $x_0 = \infty$, $q^w(x_0) \geq v - A$: by definition $A \geq F_i(x_0)\left(v - q^w(x_0) + q^l(x_0)\right) - q^l(x_0) = v - q^w(x_0)$ since $F(x_0) = 1$, meaning that $q^w(x_0) \geq v - A$. For completeness we also show that $v - A \geq 0$, by showing that $v \geq A \geq 0$: observe that $A = F_i(a)v - F_i(a)q^w(a) - (1-F_i(a))q^l(a) \leq v$, since F_i is a CDF; moreover $A \geq F_i(0)\left(v - q^w(0) + q^l(0)\right) - q^l(0) = F_i(0)v \geq 0$.

In case $\theta < 1$, $A + \sum_k p_{kj}(\mathbf{B}) \geq A + \frac{(A+\theta(v-A))(\ln(A+\theta(v-A))-\ln(v))-(\theta-1)(v-A)}{(\theta-1)^2}$, which is minimized for $A = \frac{v(\theta\cdot e^{1-\theta}-1)}{(\theta-1)e^{1-\theta}}$. The lemma follows by replacing A with this value.

In case $\theta = 1$, $A + \sum_k p_{kj}(\mathbf{B}) \geq A + \frac{(v-A)^2}{2v} \geq \frac{1}{2}v$. The limit of $\frac{\theta^2-\theta+1-e^{\theta-1}}{(\theta-1)^2}$ when $\theta \to 1$ is $\frac{1}{2}$. ∎

In the following, let $f_i^S(\cdot)$ be a maximizing additive function of set S for player i with fractionally subadditive valuation function v_i. By the definition of fractionally subadditive valuations, we have that $v_i(T) \geq f_i^S(T)$, for every $T \subseteq S$ and $f_i^S(S) = v_i(S)$.

Lemma 6.2. *For any set S of items, and any strategy profile $\mathbf{b}$, where $b_{ij} = 0$ for $j \notin S$,*

$$u_i(\mathbf{b}) \geq \sum_{j\in S}\left(\mathbb{P}[j\in X_i(\mathbf{b})]\left(f_i^S(j) - q_j^w(b_{ij}) + q_j^l(b_{ij})\right) - q_j^l(b_{ij})\right).$$

Proof.

$$
\begin{aligned}
u_i(\mathbf{b}) &\geq \sum_{T \subseteq S} \mathbb{P}[X_i(\mathbf{b}) = T] \left(f_i^S(T) - \sum_{j \in T} q_j^w(b_{ij}) - \sum_{j \in S \setminus T} q_j^l(b_{ij}) \right) \\
&= \sum_{T \subseteq S} \sum_{j \in T} \mathbb{P}[X_i(\mathbf{b}) = T] \left(f_i^S(j) - q_j^w(b_{ij}) \right) - \sum_{T \subseteq S} \sum_{j \in S \setminus T} \mathbb{P}[X_i(\mathbf{b}) = T] q_j^l(b_{ij}) \\
&= \sum_{j \in S} \sum_{T \subseteq S : j \in T} \mathbb{P}[X_i(\mathbf{b}) = T] \left(f_i^S(j) - q_j^w(b_{ij}) \right) - \sum_{j \in S} \sum_{T \subseteq S : j \notin T} \mathbb{P}[X_i(\mathbf{b}) = T] q_j^l(b_{ij}) \\
&= \sum_{j \in S} \mathbb{P}[j \in X_i(\mathbf{b})] \left(f_i^S(j) - q_j^w(b_{ij}) \right) - \sum_{j \in S} \mathbb{P}[j \notin X_i(\mathbf{b})] q_j^l(b_{ij}) \\
&= \sum_{j \in S} \Big(\mathbb{P}[j \in X_i(\mathbf{b})] (f_i^S(j) - q_j^w(b_{ij}) + q_j^l(b_{ij})) - q_j^l(b_{ij}) \Big).
\end{aligned}
$$

□

Lemma 6.3. *Let $\mathbf{B}$ be a coarse correlated equilibrium of a simultaneous bid-dependent auction. For any set of items S and any pure strategy b_i' of player i, where $b_{ij}' = 0$ for $j \notin S$,*

$$
u_i(\mathbf{B}) = \mathop{\mathbb{E}}_{\mathbf{b} \sim \mathbf{B}}[u_i(\mathbf{b})] \geq \sum_{j \in S} \Big(F_{ij}(b_{ij}') \left(f_i^S(j) - q_j^w(b_{ij}') + q_j^l(b_{ij}') \right) - q_j^l(b_{ij}') \Big).
$$

The proof of the lemma is analogous to that of Lemma 3.2: $\mathbb{E}_{\mathbf{b} \sim \mathbf{B}}[u_i(\mathbf{b})] \geq \mathbb{E}_{\mathbf{b} \sim \mathbf{B}}[u_i((b_i', \mathbf{b}_{-i}))]$ holds since $\mathbf{B}$ is an equilibrium; then Lemma 6.2 is applied to $(b_i', \mathbf{b}_{-i})$, and expectation is taken over $\mathbf{b} \sim \mathbf{B}$.

Theorem 6.4. *For bidders with fractionally subadditive valuations, the coarse correlated PoA of any bid-dependent auction is at most $\frac{(\theta-1)^2}{\theta^2-\theta+1-e^{\theta-1}}$.*

Proof. Let $\mathbf{B}$ be a coarse correlated equilibrium, and let $\lambda(\theta) = \frac{\theta^2-\theta+1-e^{\theta-1}}{(\theta-1)^2}$. For every player i, consider the maximizing additive valuation, $f_i^{O_i}$ for his optimal set O_i. By Lemma 6.1, for every fixed player i and item j there exists a bid a_{ij} such that

$$
F_{ij}(a_{ij}) \left(f_i^{O_i}(j) - q_j^w(a_{ij}) + q_j^l(a_{ij}) \right) - q_j^l(a_{ij}) \geq \lambda(\theta) f_i^{O_i}(j) - \sum_k p_{kj}(\mathbf{B})
$$

For player i, we consider the deviation that her bid is a_{ij} for every item in O_i (and 0 for all other items), and apply Lemma 6.3. Combined with the above inequality (for all items in O_i), we obtain

$$
u_i(\mathbf{B}) \geq \lambda(\theta) \sum_{j \in O_i} f_i^{O_i}(j) - \sum_{j \in O_i} \sum_k p_{kj}(\mathbf{B}) = \lambda(\theta) v_i(O_i^{\mathbf{v}}) - \sum_{j \in O_i} \sum_k p_{kj}(\mathbf{B}).
$$

By summing over all players, we get

$$
\sum_i u_i(\mathbf{B}) \geq \lambda(\theta) \sum_i v_i(O_i^{\mathbf{v}}) - \sum_{j \in [m]} \sum_k p_{kj}(\mathbf{B}) = \lambda(\theta) SW(\mathbf{O}) - \sum_k p_k(\mathbf{B})
$$

The theorem follows from $SW(\mathbf{B}) = \sum_i u_i(\mathbf{B}) + \sum_i p_i(\mathbf{B})$. □

Similarly to Lemmas 3.2 and 6.3, we can prove the following.

Lemma 6.5. *Assume that* $\mathbf{B}$ *be is a Bayesian Nash equilibrium, and let* S *be an arbitrary set of items. For player* i *with valuation* v_i*, let* b'_i *be a pure strategy such that* $b'_{ij} = 0$ *for* $j \notin S$*. Then,*

$$\mathop{\mathbb{E}}_{\substack{\mathbf{v}_{-i} \\ \mathbf{b}\sim\mathbf{B}(\mathbf{v})}} [u_i^{v_i}(\mathbf{b})] \geq \sum_{j\in S} \left(F_{ij}^{v_i}(b'_{ij})(f_{v_i}^S(j) - q_j^w(b'_{ij}) + q_j^l(b'_{ij})) - q_j^l(b'_{ij}) \right).$$

Theorem 6.6. *The Bayesian PoA of any bid-dependent auction, when the bidders have fractionally subadditive and independently distributed valuations, is at most* $\frac{(\theta-1)^2}{\theta^2-\theta+1-e^{\theta-1}}$.

Proof. Let $\lambda(\theta) = \frac{\theta^2-\theta+1-e^{\theta-1}}{(\theta-1)^2}$. Suppose that $\mathbf{B}$ is a Bayesian Nash Equilibrium and the valuation of each player i is drawn according to $v_i \sim D_i$, where the D_i are independently distributed. We use the notation $\mathbf{C} = (C_1, C_2, \ldots, C_n)$ to denote the bidding distribution in $\mathbf{B}$ which involves the randomness of the valuations $\mathbf{v}$, and of the bidding strategy $\mathbf{B}(v)$, that is $b_i(v_i) \sim C_i$. Then the utility of player i with valuation v_i can be expressed by $u_i(\mathbf{B}_i(v_i), \mathbf{C}_{-i}) = \mathbb{E}_{b_i\sim\mathbf{B}_i(v_i), \mathbf{b}_{-i}\sim\mathbf{C}_{-i}}[u_i(\mathbf{b})]$. It should be noted that $\mathbf{C}_{-i}$ does not depend on a particular v_{-i} (just on the distribution $\mathbf{D}$). Also notice that the following equality holds: $\mathbb{E}_{\mathbf{v}_{-i}}[u_i^{v_i}(\mathbf{B}_i(v_i), \mathbf{B}_{-i}(\mathbf{v}_{-i}))] = u_i^{v_i}(\mathbf{B}_i(v_i), \mathbf{C}_{-i})$ [10].

For any player i and any fractionally subadditive valuation $v_i \in V_i$, consider the following deviation: consider some $\mathbf{v}'_{-i} \sim D_{-i}$ and then for every $j \in O(v_i, \mathbf{v}'_{-i})$ bid $a_j(v_i, \mathbf{C}_{-i})$ as defined in Lemma 6.1. By applying Lemma 6.5 for $S = O_i(v_i, \mathbf{v}'_{-i})$, taking expectation over v_i and $\mathbf{v}'_{-i}$ and summing over all players, we have that

$$\begin{aligned}
&\sum_i \mathop{\mathbb{E}}_{\mathbf{v}}[u_i^{v_i}(\mathbf{B}(\mathbf{v}))] \\
&= \sum_i \mathop{\mathbb{E}}_{\mathbf{v}}[u_i^{v_i}(\mathbf{B}_i(v_i), \mathbf{C}_{-i})] \\
&\geq \sum_i \mathop{\mathbb{E}}_{v_i, \mathbf{v}'_{-i}} \left[\sum_{j\in O_i(v_i,\mathbf{v}'_{-i})} \left(F_{ij}^{v_i}(a_{ij}) \left(f_{v_i}^{O_i(v_i,\mathbf{v}'_{-i})}(j) - q_j^w(a_{ij}) + q_j^l(a_{ij}) \right) - q_j^l(a_{ij}) \right) \right] \\
&= \sum_i \mathop{\mathbb{E}}_{\mathbf{v}'} \left[\sum_{j\in O_i(\mathbf{v}')} \left(F_{ij}^{v'_i}(a_{ij}) \left(f_{v'_i}^{O_i(\mathbf{v}')}(j) - q_j^w(a_{ij}) + q_j^l(a_{ij}) \right) - q_j^l(a_{ij}) \right) \right] \\
&\geq \sum_i \mathop{\mathbb{E}}_{\mathbf{v}'} \left[\sum_{j\in O_i(\mathbf{v}')} \left(\lambda(\theta) \cdot f_{v'_i}^{O_i(\mathbf{v}')}(j) - \sum_k p_{kj}(\mathbf{B}_i(v_i), \mathbf{C}_{-i}) \right) \right] \\
&= \lambda(\theta) \cdot \sum_i \mathop{\mathbb{E}}_{\mathbf{v}}[v_i(O_i^{\mathbf{v}})] - \sum_i \sum_j p_{kj}(\mathbf{C})
\end{aligned}$$

The last inequality follows by Lemma 6.1.
So, $\mathbb{E}_{\mathbf{v}}[SW(\mathbf{B}(\mathbf{v}))] = \sum_i \mathbb{E}_{\mathbf{v}}[u_i(\mathbf{B})] + \sum_i \sum_j p_{kj}(\mathbf{C}) \geq \lambda(\theta) \cdot \mathbb{E}_{\mathbf{v}}[SW(\mathbf{O}^{\mathbf{v}})]$. □

6.1.2 Lower Bound

Here we present a lower bound of $\frac{e}{e-1}$ for the PoA of simultaneous bid-dependent auctions with OXS valuations and for mixed equilibria. This implies a lower bound for submodular and fractionally subadditive valuations, as well as for more general classes of equilibria.

[10] $\mathbb{E}_{\mathbf{v}_{-i}}[u_i^{v_i}(\mathbf{B}(\mathbf{v}))] = \mathbb{E}_{b_i\sim B_i(v_i)} \mathop{\mathbb{E}}_{\substack{\mathbf{v}_{-i} \\ \mathbf{b}_{-i}\sim\mathbf{B}_{-i}(\mathbf{v}_{-i})}} [u_i^{v_i}(b_i, b_{-i})] = \mathbb{E}_{b_i\sim B_i(v_i)} \mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{C}_{-i}}[u_i^{v_i}(b_i, b_{-i})] = u_i^{v_i}(\mathbf{B}_i(v_i), \mathbf{C}_{-i})$

Theorem 6.7. *The PoA of simultaneous bid-dependent auctions with full information and OXS valuations is at least* $\frac{e}{e-1} \approx 1.58$.

Proof. The proof is very similar to the one for simultaneous first price auctions (Section 3.2). Therefore, here we only point out the differences. The same construction applies here; the only difference appears in the Nash strategy profile and in a scaling of the valuations.

We choose an appropriate value V, such that $V\left(1-\left(\frac{n-1}{n}\right)^{n-1}\right)$ is in the range of $q_j^w(\cdot)$ for all j (notice that due to our assumptions on q_j^w, there exists such a V). We consider the same set of players and items as in Section 3.2; the valuation functions of the players are the same as in Section 3.2, except that each valuation is multiplied by V. Also, the same tie breaking rule applies.

As for the mixed Nash equilibrium $\mathbf{B}$, the dummy player still bids 0 for every item and every real player still picks an $n-1$ dimensional slice in the same random way. However the bid x_j that she bids for every item j of that slice is drawn according to a distribution with the following *item-specific* CDF (we will show below that G_j is a valid CDF):

$$G_j(x) = n\left(\frac{V\left(\frac{n-1}{n}\right)^{n-1} + q_j^l(x)}{V - q_j^w(x) + q_j^l(x)}\right)^{\frac{1}{n-1}} - n + 1, \qquad x \in [0, T_j]$$

where T_j is the bid such that $q_j^w(T_j) = V\left(1-\left(\frac{n-1}{n}\right)^{n-1}\right)$. Notice that we can no longer require that the bids of a player on different items are equal, since the CDFs G_j are different. Instead, we require that for every real player the bids x_j for different items in her slice are correlated in the following way: she chooses ρ uniformly at random from the interval $[0,1]$, and then sets $x_j = G_j^{-1}(\rho)$, for every j in her slice. Note that for any two items j_1, j_2 of the slice, it holds that $G_{j_1}(x_1) = G_{j_2}(x_2) = \rho$ and x_{j_1} is not necessarily equal to x_{j_2}. However, for each item j in the slice, the way that x_j is chosen is equivalent to sampling it according to the CDF $G_j(x_j)$ (but in a correlated way to the other bids). The fact that each player's bids are such that the CDF values become equal, will be sufficient for proving that $\mathbf{B}$ is a mixed Nash equilibrium.

The probability $F_j(x)$ that a player gets item j if she bids x for it is:

$$F_j(x) = \left(\frac{G_j(x)}{n} + \frac{n-1}{n}\right)^{n-1} = \frac{V\left(\frac{n-1}{n}\right)^{n-1} + q_j^l(x)}{V - q_j^w(x) + q_j^l(x)}, \quad x \in [0, T_j]$$

Recall that the valuation of player i is additive, restricted to the slice of items that she bids for in a particular b_i. Therefore the expected utility of i when he bids x in b_i for item j is $F_j(x)(V - q_j^w(x)) - (1-F_j(x))q_j^l(x) = F_j(x)(V - q_j^w(x) + q_j^l(x)) - q_j^l(x) = V\left(\frac{n-1}{n}\right)^{n-1}$. By comprising all items, $\mathbb{E}[u_i(b_i)] = Vn^{n-1}\left(\frac{n-1}{n}\right)^{n-1}$.

Claim 6.8. $\mathbf{B}$ is a Nash equilibrium.

Proof. First, we fix an arbitrary $w_{-i} \in [n]^{n-1}$, and focus on the set of items $C := \{(\ell, w_{-i}) \mid \ell \in [n]\}$, which we call a *column* for player i. Recall that i is interested in getting only one item within C, on the other hand his valuation is additive over items from different columns. Moreover, in a fixed $\mathbf{b}_{-i}$, every other player k submits bids x_j resulting in equal values of $G_j(x_j)$ for all items in C, because either the whole C is in the current slice of k, and he bids correlated bids on them, or no item from the column is in the slice and he bids 0.

Consider first a deviating bid, in which i bids a positive value for more than one items in C, say (at least) the values $x_1, x_2 > 0$ for items j_1, j_2, respectively, and w.l.o.g. assume that $G_{j_1}(x_1)$ is maximum over items in C. We prove that if she loses item j_1 she should lose item j_2 as well: if she loses j_1, then there must be a bidder k with bid $x'_1 > x_1$ for item j_1. Since $G_{j_1}(x)$ is increasing, this implies $G_{j_1}(x'_1) > G_{j_1}(x_1)$. However, since the bids of player k are correlated (and j_2 is in his slice as well), for his bid x'_2 on j_2 it holds that $G_{j_2}(x'_2) = G_{j_1}(x'_1) > G_{j_1}(x_1) \geq G_{j_2}(x_2)$. Therefore, $x'_2 > x_2$, so player i cannot win item j_2 either, so bidding for item j_2 cannot contribute to the valuation, whereas the bidder might pay for more items than j_1. Consequently, bidding for only one item in C and 0 for the rest of C is more profitable.

Second, observe that restricted to a fixed column, submitting any bid $x \in [0, T_j]$ for one arbitrary item j results in the constant expected utility of $V\left(\frac{n-1}{n}\right)^{n-1}$, whereas by bidding higher than T_j the utility would be at most $V - q^w_j(T_j) = V\left(\frac{n-1}{n}\right)^{n-1}$ for this column. In summary, bidding for exactly one item j from each column, an arbitrary bid $x \in [0, T_j]$ is a best response for i yielding the above expected utility, which concludes the proof that $\mathbf{B}$ is a Nash equilibrium. □

The rest of the argument is exactly the same as in the proof for first price auctions, with both $SW(\mathbf{B})$ and $SW(\mathbf{O})$ scaled by V, that cancels out in the price of anarchy.

It remains to prove that the $G_j(\cdot)$ are valid cumulative distribution functions for every j. To this end it is sufficient to show that $G_j(T_j) = 1$ and that $G_j(x)$ is non-decreasing in $[0, T_j]$. For simplicity we skip index j.

$$G(T) = n\left(\frac{V\left(\frac{n-1}{n}\right)^{n-1} + q^l(T)}{V - q^w(T) + q^l(T)}\right)^{\frac{1}{n-1}} - n + 1 =$$

$$= n\left(\frac{V\left(\frac{n-1}{n}\right)^{n-1} + q^l(T)}{V - V\left(1 - \left(\frac{n-1}{n}\right)^{n-1}\right) + q^l(T)}\right)^{\frac{1}{n-1}} - n + 1 = 1$$

Now let $x_1, x_2 \in [0, T_j]$, and $x_1 > x_2$. In order to prove $G(x_1) \geq G(x_2)$, it is sufficient to prove that $\frac{V\left(\frac{n-1}{n}\right)^{n-1} + q^l(x_1)}{V - q^w(x_1) + q^l(x_1)} \geq \frac{V\left(\frac{n-1}{n}\right)^{n-1} + q^l(x_2)}{V - q^w(x_2) + q^l(x_2)}$.

$$\begin{aligned}
& \frac{V\left(\frac{n-1}{n}\right)^{n-1} + q^l(x_1)}{V - q^w(x_1) + q^l(x_1)} - \frac{V\left(\frac{n-1}{n}\right)^{n-1} + q^l(x_2)}{V - q^w(x_2) + q^l(x_2)} \\
= \quad & \frac{q^l(x_1)(V - V\left(\frac{n-1}{n}\right)^{n-1} - q^w(x_2)) - q^l(x_2)(V - V\left(\frac{n-1}{n}\right)^{n-1} - q^w(x_1))}{(V - q^w(x_1) + q^l(x_1))(V - q^w(x_2) + q^l(x_2))} \\
\geq \quad & \frac{(q^l(x_1) - q^l(x_2))(V - V\left(\frac{n-1}{n}\right)^{n-1} - q^w(x_1))}{(V - q^w(x_1) + q^l(x_1))(V - q^w(x_2) + q^l(x_2))} \geq 0
\end{aligned}$$

The last two inequalities follow from the monotonicity of q^l and q^w, and from the fact that $V\left(1 - \left(\frac{n-1}{n}\right)^{n-1}\right) \geq q^w(x_1)$ holds by the definition of T_j. □

6.2 Subadditive valuations

We prove tight bounds for the PoA in simultaneous bid-dependent auctions with subadditive bidder valuations. We show that the coarse-correlated and the Bayesian PoA is exactly 2. Our results hold even for a class of auctions more general than bid-dependent auctions: we allow the payment rule to depend on the *rank* of the bid, where the r^{th} highest bid (with an arbitrary tie-breaking rule) has rank r. We use $q_j(x, r)$ to denote the payment that the bidder should pay for item j when her bid is x and gets rank r. In particular, given a mixed bidding strategy $\mathbf{B}$, bidder i's expected payment

$p_i(\mathbf{B})$ is equal to $\sum_{j\in[m]} \mathbb{E}_{\mathbf{b}}[q_j(b_{ij}, r_i(\mathbf{b}^j)]$ where $r_i(\mathbf{b}^j)$ denotes the rank of b_{ij} among $\{b_{1j}, \ldots, b_{nj}\}$. That is, $p_{ij}(\mathbf{B}) = \mathbb{E}_{\mathbf{b}}[q_j(b_{ij}, r_i(\mathbf{b}^j)]$. Note that $q_j^w(x)$ from the previous subsection is $q_j(x,1)$ here, and $q_j^l(x)$ can be different for different ranks. Analogous assumptions to the ones made on $q_j^w(x)$ and $q_j^l(x)$ can be made on $q_j(x,r)$ as well. For the following upper bound we only assume that the $q_j(.,r)$ are normalized and increasing, and $q_j(x,1) \geq q_j(x,r)$.

6.2.1 Upper Bounds

Lemma 6.9. *For any simultaneous bid-dependent auction, subadditive valuation profile* $\mathbf{v}$ *and randomized bidding profile* $\mathbf{B}$*, there exists a randomized bid vector* $A_i(\mathbf{v}, \mathbf{B}_{-i})$ *for each player* i*, such that for the total expected utility and expected payments of the bidders*

$$\sum_i u_i(A_i(\mathbf{v}, \mathbf{B}_{-i}), \mathbf{B}_{-i}) \geq \frac{1}{2}\sum_i v_i(O_i^{\mathbf{v}}) - \sum_i \sum_j p_{ij}(\mathbf{B})$$

holds, where $O_i^{\mathbf{v}}$ *is the optimal set of player* i*.*

Proof. Under the profile $\mathbf{v}$, $O_i^{\mathbf{v}}$ is the set of items allocated to player i in the optimum. We denote by $h_j(\mathbf{b}) = \arg\max_i b_{ij}$ the bidder with the highest bid for item j, regarding the pure bidding $\mathbf{b}$. Let t_{ij} be the maximum of bids for item j among players other than i, and t_i be the vector such that its j^{th} coordinate equals t_{ij} if $j \in O_i^{\mathbf{v}}$, and 0 otherwise. Note that $t_i \sim T_i$ is an induced random variable of $\mathbf{B}_{-i}$. We define the randomized bid $A_i(\mathbf{v}, \mathbf{B}_{-i})$ to follow the same distribution T_i (inspired by [16]).

We use the notation $v_i(b_i, t_i)$ and $W_i(b_i, t_i)$ to denote the player i's valuation and winning set when she bids b_i and the prices are t_i, i.e., $v_i(S)$ and $W_i(S)$ where $S = \{j | b_{ij} \geq t_{ij}\}$.

$$\begin{aligned}
& u_i(A_i(\mathbf{v}, \mathbf{B}_{-i}), \mathbf{B}_{-i}) \\
= \quad & \mathop{\mathbb{E}}_{a_i\sim A_i} \mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim \mathbf{B}_{-i}} [u_i(a_i, \mathbf{b}_{-i})] \\
\geq \quad & \mathop{\mathbb{E}}_{a_i\sim T_i} \mathop{\mathbb{E}}_{t_i\sim T_i} [v_i(a_i, t_i)] - \sum_{j\in O_i^{\mathbf{v}}} \mathop{\mathbb{E}}_{a_i\sim T_i}[q_j(a_{ij}, 1)] \qquad \text{(since } q_j(x,1) \geq q_j(x,r)\text{)} \\
= \quad & \mathop{\mathbb{E}}_{t_i\sim T_i} \mathop{\mathbb{E}}_{a_i\sim T_i} [v_i(t_i, a_i)] - \sum_{j\in O_i^{\mathbf{v}}} \mathop{\mathbb{E}}_{t_i\sim T_i}[q_j(t_{ij}, 1)] \qquad \text{(swap } t_i \text{ and } a_i\text{)} \\
= \quad & \frac{1}{2} \mathop{\mathbb{E}}_{t_i\sim T_i} \mathop{\mathbb{E}}_{a_i\sim T_i} [v_i(t_i, a_i) + v_i(a_i, t_i)] - \sum_{j\in O_i^{\mathbf{v}}} \mathop{\mathbb{E}}_{t_i\sim T_i}[q_j(t_{ij}, 1)] \\
\geq \quad & \frac{1}{2} v_i(O_i^{\mathbf{v}}) - \sum_{j\in O_i^{\mathbf{v}}} \mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[q_j(b_{h_j(\mathbf{b})}(j), 1)] \\
\geq \quad & \frac{1}{2} v_i(O_i^{\mathbf{v}}) - \sum_{j\in O_i^{\mathbf{v}}} \mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[q_j(b_{h_j(\mathbf{b})}(j), 1)] \\
- \quad & \sum_{j\in O_i^{\mathbf{v}}} \mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}} \left[\sum_k q_j(b_k(j), r_k(\mathbf{b}^j)) - q_j(b_{h_j(\mathbf{b})}(j), r_{h_j(\mathbf{b})}(\mathbf{b}^j))\right] \\
= \quad & \frac{1}{2} v_i(O_i^{\mathbf{v}}) - \sum_{j\in O_i^{\mathbf{v}}} \sum_k \mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[q_j(b_k, r(k, \mathbf{b}))] = \frac{1}{2} v_i(O_i^{\mathbf{v}}) - \sum_{j\in O_i^{\mathbf{v}}} \sum_k p_{kj}(\mathbf{B})
\end{aligned}$$

In the second inequality $v_i(t_i, a_i) + v_i(a_i, t_i) \geq v_i(O_i^{\mathbf{v}})$ is due to the subadditivity of v_i; for $q_j(t_{ij}, 1) \leq q_j(b_{h_j(\mathbf{b})}(j), 1)$ we use that $q_j(., 1)$ is non-decreasing, and the fact that $t_{ij} \leq b_{h_j(\mathbf{b})}(j)$, since in

$b_{h_j(\mathbf{b})}(j)$, for computing the maximum we also consider player i. For the last inequality, notice that $\sum_k q_j(b_k(j), r_k(\mathbf{b}^j)) - q_j(b_{h_j(\mathbf{b})}(j), r_{h_j(\mathbf{b})}(\mathbf{b}^j)) \geq 0$, since from the sum of all payments for item j we subtracted the payment of the winner. The lemma follows by summing over all players. $\square$

Theorem 6.10. *For bidders with subadditive valuations, the coarse correlated PoA of any bid-dependent auction is at most* 2*.*

Proof. Suppose $\mathbf{B}$ is a coarse correlated equilibrium (notice that $\mathbf{v}$ is fixed). By Lemma 6.9 and the definition of coarse correlated equilibrium, we have

$$\sum_i u_i(\mathbf{B}) \geq \sum_i u_i(A_i(\mathbf{v}, \mathbf{B}_{-i}), \mathbf{B}_{-i}) \geq \frac{1}{2} \sum_i v_i(O_i^{\mathbf{v}}) - \sum_i \sum_j p_{ij}(\mathbf{B}).$$

By rearranging the terms, $SW(B) = \sum_i u_i(\mathbf{B}) + \sum_i \sum_j p_{ij}(\mathbf{B}) \geq 1/2 \cdot SW(\mathbf{O})$. $\square$

Theorem 6.11. *For bidders with subadditive and independent valuations, the Bayesian PoA of any bid-dependent auction is at most* 2*.*

Proof. Suppose $\mathbf{B}$ is a Bayesian Nash Equilibrium and the valuation of each player i is $v_i \sim D_i$, where the D_i are independently distributed. We use the notation $\mathbf{C} = (C_1, C_2, \ldots, C_n)$ to denote the bidding distribution in $\mathbf{B}$ which includes the randomness of the valuations $\mathbf{v}$, and of the bidding strategy $\mathbf{b}$, that is $b_i(v_i) \sim C_i$ (like in the proof of Theorem 6.6). Then the utility of player i with valuation v_i can be expressed by $u_i(B_i(v_i), \mathbf{C}_{-i})$. For any player i and any subadditive valuation $v_i \in V_i$, consider the following deviation: sampling $\mathbf{v}'_{-i} \sim D_{-i}$ and bidding $A_i((v_i, \mathbf{v}'_{-i}), \mathbf{C}_{-i})$ as defined in Lemma 6.9. By the definition of Nash equilibrium, we have $\mathbb{E}_{\mathbf{v}_{-i}}[u_i^{v_i}(B_i(v_i), \mathbf{B}_{-i}(\mathbf{v}_{-i}))] \geq \mathbb{E}_{\mathbf{v}'_{-i}}[u_i^{v_i}(A_i((v_i, \mathbf{v}'_{-i}), \mathbf{C}_{-i}), \mathbf{C}_{-i})]$. By taking expectation over v_i and summing over all players,

$$\begin{aligned}
\sum_i \mathbb{E}_{\mathbf{v}}[u_i(\mathbf{B}(\mathbf{v}))] &\geq \sum_i \mathbb{E}_{v_i, \mathbf{v}'_{-i}}[u_i^{v_i}(A_i((v_i, \mathbf{v}'_{-i}), \mathbf{C}_{-i}), \mathbf{C}_{-i})] \\
&= \mathbb{E}_{\mathbf{v}}\left[\sum_i u_i^{v_i}(A_i(\mathbf{v}, \mathbf{C}_{-i}), \mathbf{C}_{-i})\right] \text{ (by relabeling } \mathbf{v}'_{-i} \text{ by } \mathbf{v}_{-i}) \\
&\geq \frac{1}{2} \cdot \sum_i \mathbb{E}_{\mathbf{v}}[v_i(O_i^{\mathbf{v}})] - \sum_i \sum_j \mathbb{E}_{\mathbf{v}}[p_{kj}(\mathbf{B}(\mathbf{v}))]
\end{aligned}$$

where the inequality follows by Lemma 6.9. We obtained $\mathbb{E}_{\mathbf{v}}[SW(\mathbf{B}(\mathbf{v}))] = \sum_i \mathbb{E}_{\mathbf{v}}[u_i(\mathbf{B})] + \sum_i \sum_j \mathbb{E}_{\mathbf{v}}[p_{kj}(\mathbf{B}(\mathbf{v}))] \geq 1/2 \cdot \mathbb{E}_{\mathbf{v}}[SW(\mathbf{O}^{\mathbf{v}})]$. $\square$

6.2.2 Lower Bound

Theorem 6.12. *For bidders with subbaditive valuations, the mixed PoA of simultaneous bid-dependent auctions is at least* 2*.*

Proof. We consider 2 players and m items. Let v and V (with $v < V$) be positive reals to be defined later. Player 1 has value v for every non-empty subset of items; player 2 values with V any non-empty strict subset of the items and with $2V$ the whole set of items. Consider now the mixed strategy profile $\mathbf{B}$, where player 1 picks item l uniformly at random and bids x_l for it and 0 for the rest of the items, whereas, player 2 bids y_j for every item j. For $1 \leq j \leq m$, x_j and y_j are drawn from distributions with the following CDFs $G_j(x)$ and $F_j(y)$, respectively:

$$G_j(x) = \frac{(m-1)q_j^w(x) + q_j^l(x)}{V - q_j^w(x) + q_j^l(x)}, \;\; x \in [0, T_j]; \qquad F_j(y) = \frac{v - V/m + q_j^l(y)}{v - q_j^w(y) + q_j^l(y)}, \;\; y \in [0, T_j],$$

where T_j is the bid such that $q_j^w(T_j) = V/m$. We choose V such that V/m is in the range of $q_j^w(\cdot)$ for all j (notice that due to the assumptions on q_j^w, there always exists such a value V). Furthermore, in $\mathbf{B}$, the y_j's are correlated in the following way: player 2 chooses ρ uniformly at random from the interval $[0,1]$ and if $\rho \in \left[0, \frac{v-V/m}{v}\right)$ then $y_j = 0$, otherwise $y_j = F_j^{-1}(\rho)$, for every $1 \leq j \leq m$.[11] Note that for every two items j_1, j_2, it holds that $F_{j_1}(y_{j_1}) = F_{j_2}(y_{j_2})$. In case of a tie, player 2 gets the item. Due to the continuity of q_j^w and q_j^l, $G_j(x)$ and $F_j(x)$ are continuous and therefore none of the CDF have a mass point in any $x \neq 0$.

We show below that $\mathbf{B}$ is a Nash equilibrium, and each of the F_j and G_j are valid cumulative distributions. The PoA can then be derived as follows. Player 2 bids 0 with probability $1 - \frac{V}{mv}$ so, $\mathbb{E}[SW(\mathbf{B})] \leq \left(1 - \frac{V}{mv}\right)(V+v) + \frac{V}{mv}2V = V + v + \frac{V^2}{mv} - \frac{V}{m}$. For $v = V/\sqrt{m}$, PoA $\geq \frac{2V}{V + \frac{2V}{\sqrt{m}} - \frac{V}{m}} = \frac{2}{1 + \frac{2}{\sqrt{m}} - \frac{1}{m}}$ which, for large m, converges to 2.

Claim 6.13. $\mathbf{B}$ is a Nash equilibrium.

Proof. If player 1 bids any x in the range of $(0, T_j]$ for a single item j and zero for the rest, her utility is $F_j(x)(v - q_j^w(x)) + (1 - F_j(x))(-q_j^l(x)) = F_j(x)(v - q_j^w(x) + q_j^l(x)) - q_j^l(x) = v - V/m$. Since $G(0) = 0$, her utility is also $v - V/m$ if she bids according to $G(\cdot)$. Suppose player 1 bids $x = (x_1, \ldots, x_m)$, ($x_j \in [0, T_j]$) with at least two positive bids. W.l.o.g., assume $F_1(x_1) = \max_i F_i(x_i)$. If $y_1 \geq x_1$, player 1 doesn't get any item, since for every j, $F_j(y_j) = F_1(y_1) \geq F_1(x_1) \geq F_j(x_j)$ and so $y_j \geq x_j$ (recall that in any tie player 2 gets the item). If $y_1 < x_1$, player 1 gets at least the first item and has valuation v, but she cannot pay less than $q_1^w(x_1)$. So, this strategy is dominated by the strategy of bidding x_1 for the first item and zero for the rest. Bidding $x_j > T_j$ for any item guarantees the item but results in a utility less than $v - q_j^w(x_j) \leq v - q_j^w(T_j) = v - V/m$, so it is dominated by the strategy of bidding exactly T_j for this item.

If player 2 bids $(y_1, \ldots, y_m)$ for every item j so that $y_j \in [0, T_j]$, then (since player 1 bids positive for any particular item j with probablility $1/m$) her expected utility is

$$
\begin{aligned}
& \frac{1}{m} \sum_{j=1}^{m} \left(G_j(y_j)(2V - \sum_{k=1}^{m} q_k^w(y_k)) + (1 - G_j(y_j))(V - \sum_{\substack{k=1 \\ k \neq j}}^{m} q_k^w(y_k) - q_j^l(y_j)) \right) \\
= \; & \frac{1}{m} \sum_{j=1}^{m} \left(V + G_j(y_j)(V - q_j^w(y_j) + q_j^l(y_j)) - \sum_{\substack{k=1 \\ k \neq j}}^{m} q_k^w(y_k) - q_j^l(y_j) \right) \\
= \; & \frac{1}{m} \sum_{j=1}^{m} \left(V + (m-1) q_j^w(y_j) + q_j^l(y_j) - \sum_{\substack{k=1 \\ k \neq j}}^{m} q_k^w(y_k) - q_j^l(y_j) \right) \\
= \; & \frac{1}{m} \left(mV + m \sum_{j=1}^{m} q_j^w(y_j) - m \sum_{k=1}^{m} q_k^w(y_k) \right) = V.
\end{aligned}
$$

Bidding greater than T_j for any item is dominated by the strategy of bidding exactly T_j for this item. Overall, $\mathbf{B}$ is Nash equilibrium. □

[11] For each item j, the way player 2 chooses y_j is equivalent to picking it according to the CDF $F_j(y)$.

Claim 6.14. $G_j(\cdot)$ and $F_j(\cdot)$ are valid cumulative distributions.

Proof. It is sufficient to show that for every j, $G_j(T_j) = F_j(T_j) = 1$ and $G_j(x)$ and $F_j(x)$ are non-decreasing in $[0, T_j]$. In the following we skip index j.

$$G(T) = \frac{(m-1)q^w(T) + q^l(T)}{V - q^w(T) + q^l(T)} = \frac{(m-1)\frac{V}{m} + q^l(T)}{V - \frac{V}{m} + q^l(T)} = 1,$$

$$F(T) = \frac{v - V/m + q^l(T)}{v - q^w(T) + q^l(T)} = \frac{v - V/m + q^l(T)}{v - V/m + q^l(T)} = 1,$$

Now let $x_1 > x_2$, $x_1, x_2 \in [0, T_j]$.

$$\begin{aligned}
& G(x_1) - G(x_2) \\
= \quad & \frac{(m-1)q^w(x_1) + q^l(x_1)}{V - q^w(x_1) + q^l(x_1)} - \frac{(m-1)q^w(x_2) + q^l(x_2)}{V - q^w(x_2) + q^l(x_2)} \\
= \quad & \frac{V(m-1)(q^w(x_1) - q^w(x_2)) + m(q^w(x_1)q^l(x_2) - q^l(x_1)q^w(x_2)) + V(q^l(x_1) - q^l(x_2))}{(V - q^w(x_1) + q^l(x_1))(V - q^w(x_2) + q^l(x_2))} \\
= \quad & \frac{(V(m-1) + m \cdot q^l(x_2))(q^w(x_1) - q^w(x_2)) + m(\frac{V}{m} - q^w(x_2))(q^l(x_1) - q^l(x_2))}{(V - q^w(x_1) + q^l(x_1))(V - q^w(x_2) + q^l(x_2))} \geq 0
\end{aligned}$$

$$\begin{aligned}
& F_j(x_1) - F_j(x_2) \\
= \quad & \frac{v - V/m + q^l(x_1)}{v - q^w(x_1) + q^l(x_1)} - \frac{v - V/m + q^l(x_2)}{v - q^w(x_2) + q^l(x_2)} \\
= \quad & \frac{(v - \frac{V}{m})(q^w(x_1) - q^w(x_2)) + \frac{V}{m}(q^l(x_1) - q^l(x_2)) + q^w(x_1)q^l(x_2) - q^l(x_1)q^w(x_2)}{(v - q^w(x_1) + q^l(x_1))(v - q^w(x_2) + q^l(x_2))} \\
= \quad & \frac{(v - \frac{V}{m} + q^l(x_2))(q^w(x_1) - q^w(x_2)) + (\frac{V}{m} - q^w(x_2))(q^l(x_1) - q^l(x_2))}{(v - q^w(x_1) + q^l(x_1))(v - q^w(x_2) + q^l(x_2))} \geq 0
\end{aligned}$$

For both inequalities we use the monotonicity of q, moreover that $q^w_j(x) \leq V/m$ for $x \in [0, T_j]$, and $v = V/\sqrt{m}$ hold. □

□

7 Discriminatory auctions

Discriminatory auctions are *multi-unit* auctions, i.e. m units of the same item are sold to n bidders. We denote the valuation of player i for j units of the item by $v_i(j)$. The valuation v_i is *submodular*, if the items have decreasing marginal values, that is, $v_i(s+1) - v_i(s) \geq v_i(t+1) - v_i(t)$ holds if $s \leq t$. It is called *subadditive*, if $v_i(s+t) \leq v_i(s) + v_i(t)$.

We assume a *standard* multi-unit auction in which each player submits a vector b_i of m decreasing bids $b_i(1) \geq b_i(2) \geq \ldots \geq b_i(m) \geq 0$. The bidding profile of all players is then $\mathbf{b} = (b_1, b_2, \ldots, b_n)$. In the allocation $\xi(\mathbf{b}) = (\xi_1, \xi_2, \ldots, \xi_n)$, bidder i gets ξ_i units of the item, if ξ_i of his bids were among the m highest bids of the players. In the case of *discriminatory pricing,* every bidder i pays the sum of his *winning* bids, i.e. his ξ_i highest bids.

In this section, we complement the results by de Keijzer et al. [12] for the case of subadditive valuations, by providing a matching lower bound of 2 for the standard bidding format. For the case of submodular valuations, we provide a lower bound of 1.109. We could reprove their upper bound of $e/(e-1)$ for submodular bids, using our non-smooth approach. Due to the different nature of this auction, the proof is not identical with the one for the first-price auction. Therefore, we present the complete proof of this upper bound.

7.1 Preliminaries

The *social welfare* of the allocation $\xi(\mathbf{b})$ is $SW(\mathbf{b}) = SW(\xi(\mathbf{b})) = \sum_{i=1}^{n} v_i(\xi_i)$. The players have quasi-linear utility functions:

$$u_i(\mathbf{b}) = v_i(\xi_i) - \sum_{j=1}^{\xi_i} b_i(j)$$

Similarly to item bidding auctions, having a *mixed strategy* B_i, means that b_i is drawn from the set of all possible decreasing bid vectors according to the distribution B_i, which we denote by $b_i \sim B_i$. Given a valuation profile $\mathbf{v}$ of the players, an *optimal allocation* $\mathbf{o}(\mathbf{v}) = \mathbf{o}^{\mathbf{v}} = (\mathbf{o}_1^{\mathbf{v}}, \dots \mathbf{o}_n^{\mathbf{v}})$ is one that maximizes $\sum_{i=1}^{n} v_i(o_i^{\mathbf{v}})$.

Consider a discriminatory auction with submodular valuations, with n players and m items. Recall that $v_i(j)$ denotes the valuation of player i for j copies of the item. For any player i, we define $v_{ij} = \frac{v_i(j)}{j}$. It is easy to see that for submodular functions, $v_{ij} \geq v_{i(j+1)}$ for all $j \in [m-1]$. Let $\beta_j(\mathbf{b})$ be the j^{th} *lowest* bid among the *winning* bids under the strategy profile $\mathbf{b}$. Consider any randomized bidding profile $\mathbf{B} = (B_1, ..., B_n)$. For this $\mathbf{B}$, $\beta_j(\mathbf{b})$ is a random variable depending on $\mathbf{b} \sim \mathbf{B}$. We define the following functions:

$$\begin{aligned} F_{ij}(x) &= \mathbb{P}[\beta_j(\mathbf{b}_{-i}) \leq x] && \text{for } 1 \leq j \leq m, \\ G_{ij}(x) &= \mathbb{P}[\beta_j(\mathbf{b}_{-i}) \leq x < \beta_{j+1}(\mathbf{b}_{-i})] = F_{ij}(x) - F_{i(j+1)}(x) && \text{for } 1 \leq j \leq m-1. \end{aligned}$$

We define separately $G_{im}(x) = \mathbb{P}[\beta_m(\mathbf{b}_{-i}) \leq x] = F_{im}(x)$. Notice that $F_{ij}(x)$ is the CDF of $\beta_j(\mathbf{b}_{-i})$; moreover for the G_{ij} holds that

$$\begin{aligned} F_{ij}(x) &= \sum_{k=j}^{m} G_{ik}(x), \\ \sum_{j=1}^{m'} F_{ij}(x) &= \sum_{j=1}^{m'} jG_{ij}(x) + \sum_{j=m'+1}^{m} m'G_{ij}(x). \end{aligned} \tag{3}$$

We further define $F_i^{\mathrm{av}}(x) = \frac{1}{o_i^{\mathbf{v}}} \sum_{j=1}^{o_i^{\mathbf{v}}} F_{ij}(x)$, and let β_i^{av} be a random variable with $F_i^{\mathrm{av}}(x)$ as CDF. $F_i^{\mathrm{av}}(x)$ is a cumulative distribution function defined on $\mathbb{R}^+$, since $F_i^{\mathrm{av}}(0) = 0$, $\lim_{x \to +\infty}(F_i^{\mathrm{av}}(x)) = 1$ and $F_i^{\mathrm{av}}(x)$ is the average of non-decreasing functions, so it is itself a non-decreasing function.

Moreover,

$$\begin{aligned}\mathbb{E}[\beta_i^{\text{av}}] &= \int_0^\infty (1 - F_i^{\text{av}}(x))dx = \int_0^\infty (1 - \frac{1}{o_i^{\mathbf{v}}} \sum_{j=1}^{o_i^{\mathbf{v}}} F_{ij}(x))dx \\ &= \frac{1}{o_i^{\mathbf{v}}} \sum_{j=1}^{o_i^{\mathbf{v}}} \int_0^\infty (1 - F_{ij}(x))\, dx = \frac{1}{o_i^{\mathbf{v}}} \sum_{j=1}^{o_i^{\mathbf{v}}} \beta_j(\mathbf{B}_{-i}),\end{aligned}$$

where $\beta_j(\mathbf{B}_{-i}) = \mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[\beta_j(\mathbf{b}_{-i})]$. Note that the above functions depend on some randomized bidding profile $\mathbf{B}_{-i}$ and on $\mathbf{v}$. These will be clear from context when we use these functions below.

7.2 Upper bound for submodular valuations

Lemma 7.1. *For any submodular valuation profile $\mathbf{v}$ and any randomized bidding profile $\mathbf{B}$, there exists a pure bidding strategy $\mathbf{a}_i(\mathbf{v}, \mathbf{B}_{-i})$ for each player i, such that:*

$$\sum_{i=1}^n u_i(\mathbf{a}_i(\mathbf{v}, \mathbf{B}_{-i}), \mathbf{B}_{-i}) \geq \left(1 - \frac{1}{e}\right) \sum_{i=1}^n v_i(o_i^{\mathbf{v}}) - \sum_{j=1}^m \beta_j(\mathbf{B}),$$

where $\beta_j(\mathbf{B}) = \mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[\beta_j(\mathbf{b})]$.

Proof. Recall that $v_{ij} = \frac{v_i(j)}{j}$. Let a_i be the value that maximizes $(v_{io_i^{\mathbf{v}}} - a_i)F_i^{\text{av}}(a_i)$. Let $\mathbf{a}_i(\mathbf{v}, \mathbf{B}_{-i}) = (\underbrace{a_i, \ldots, a_i}_{o_i^{\mathbf{v}}}, \underbrace{0, \ldots, 0}_{m - o_i^{\mathbf{v}}})$ be the selected strategy profile for player i. Observe that by the definition of $G_{ij}()$, $G_{ij}(a_i)$ is the probability of a_i being the j^{th} lowest bid among winning bids under $\mathbf{B}_{-i}$. Therefore, if player i bids according to $\mathbf{a}_i(\mathbf{v}, \mathbf{B}_{-i})$, $G_{ij}(a_i)$ is the probability of player i getting exactly j items, if $j \leq o_i^{\mathbf{v}}$, and $o_i^{\mathbf{v}}$ items, if $j > o_i^{\mathbf{v}}$, under the bidding profile $(\mathbf{a}_i(\mathbf{v}, \mathbf{B}_{-i}), \mathbf{B}_{-i})$. Similarly to Lemma 3.2, we get

$$\begin{aligned}u_i(\mathbf{a}_i(\mathbf{v}, \mathbf{B}_{-i}), \mathbf{B}_{-i}) &\geq \sum_{j=1}^{o_i^{\mathbf{v}}} G_{ij}(a_i)(v_i(j) - j a_i) + \sum_{j=o_i^{\mathbf{v}}+1}^{m} G_{ij}(a_i)(v_i(o_i^{\mathbf{v}}) - o_i^{\mathbf{v}} a_i) \\ &= \sum_{j=1}^{o_i^{\mathbf{v}}} jG_{ij}(a_i)(v_{ij} - a_i) + \sum_{j=o_i^{\mathbf{v}}+1}^{m} o_i^{\mathbf{v}} G_{ij}(a_i)(v_{io_i^{\mathbf{v}}} - a_i) \\ &\geq (v_{io_i^{\mathbf{v}}} - a_i)\left(\sum_{j=1}^{o_i^{\mathbf{v}}} jG_{ij}(a_i) + \sum_{j=o_i^{\mathbf{v}}+1}^{m} o_i^{\mathbf{v}} G_{ij}(a_i)\right) \\ &= (v_{io_i^{\mathbf{v}}} - a_i) \sum_{j=1}^{o_i^{\mathbf{v}}} F_{ij}(a_i) = o_i^{\mathbf{v}}(v_{io_i^{\mathbf{v}}} - a_i)F_i^{\text{av}}(a_i) \\ &\geq \left(1 - \frac{1}{e}\right) o_i^{\mathbf{v}} v_{io_i^{\mathbf{v}}} - o_i^{\mathbf{v}}\, \mathbb{E}\left[\beta_i^{\text{av}}\right] = \left(1 - \frac{1}{e}\right) v_i(o_i^{\mathbf{v}}) - o_i^{\mathbf{v}}\, \mathbb{E}\left[\beta_i^{\text{av}}\right] \\ &= \left(1 - \frac{1}{e}\right) v_i(o_i^{\mathbf{v}}) - \sum_{j=1}^{o_i^{\mathbf{v}}} \beta_j(\mathbf{B}_{-i})\end{aligned}$$

For the second inequality, $v_{ij} \geq v_{io_i}$ for submodular valuations and for the following equality, we used (3) where m' is set to $o_i^{\mathbf{v}}$. For the last inequality we apply Lemma 3.3, since a_i maximizes the expression $(v_{io_i^{\mathbf{v}}} - a_i)F_i^{\text{av}}(a_i)$.

For any pure strategy profile $\mathbf{b}$ and any valuation profile $\mathbf{v}$ it holds that

$$\sum_{j=1}^{m} \beta_j(\mathbf{b}) \geq \sum_{i=1}^{n} \sum_{j=1}^{o_i^{\mathbf{v}}} \beta_j(\mathbf{b}) \geq \sum_{i=1}^{n} \sum_{j=1}^{o_i^{\mathbf{v}}} \beta_j(\mathbf{b}_{-i}).$$

By summing up over all players and using this inequality the lemma follows. $\square$

Theorem 7.2. *The coarse correlated PoA for the discriminatory auction is at most $\frac{e}{e-1}$, when the players' valuations are submodular.*

Proof. Suppose $\mathbf{B}$ is a coarse correlated equilibrium (in this case $\mathbf{v}$ is fixed). By Lemma 7.1 and the definition of coarse correlated equilibrium, we have that

$$\begin{aligned} \sum_{i=1}^{n} u_i(\mathbf{B}) &\geq \sum_{i=1}^{n} u_i(\mathbf{a}_i(\mathbf{v}, \mathbf{B}_{-i}), \mathbf{B}_{-i}) \\ &\geq \left(1 - \frac{1}{e}\right) \sum_{i=1}^{n} v_i(o_i^{\mathbf{v}}) - \sum_{j=1}^{m} \beta_j(\mathbf{B}) \end{aligned}$$

After rearranging the terms $SW(\mathbf{B}) = \sum_i u_i(\mathbf{B}) + \sum_j \beta_j(\mathbf{B}) \geq \left(1 - \frac{1}{e}\right) SW(\mathbf{o})$. $\square$

Theorem 7.3. *The BPoA of the discriminatory auction is at most $\frac{e}{e-1}$, when the players' valuations are submodular.*

Proof. Suppose $\mathbf{B}$ is a Bayesian Nash Equilibrium and the valuation of each player i is $v_i \sim D_i$, where the D_i's are independent distributions. We denote by $\mathbf{C} = (C_1, C_2, \ldots, C_n)$ the bidding distribution in $\mathbf{B}$ which includes the randomness of both the bidding strategy $\mathbf{b}$ and of the valuations $\mathbf{v}$, that is $b_i(v_i) \sim C_i$. Then the utility of agent i with valuation v_i can be expressed by $u_i(\mathbf{B}_i(v_i), \mathbf{C}_{-i})$. It should be noted that $\mathbf{C}_{-i}$ depends on D_{-i} but *not* on the $\mathbf{v}_{-i}$. For any agent i and any submodular valuation $v_i \in V_i$, consider the following deviation: sample $\mathbf{v}'_{-i} \sim \mathbf{D}_{-i}$ and bid $a_i((v_i, \mathbf{v}'_{-i}), \mathbf{C}_{-i})$ as defined in Lemma 7.1. By the definition of the Bayesian Nash equilibrium, we have

$$\mathop{\mathbb{E}}_{\mathbf{v}_{-i}}[u_i^{v_i}(\mathbf{B}_i(v_i), \mathbf{B}_{-i}(\mathbf{v}_{-i}))] \geq \mathop{\mathbb{E}}_{\mathbf{v}'_{-i}}[u_i^{v_i}(a_i((v_i, \mathbf{v}'_{-i}), \mathbf{C}_{-i}), \mathbf{C}_{-i})]$$

By taking expectation over v_i and summing up over all agents,

$$\begin{aligned}
&\sum_{i=1}^{n} \mathbb{E}_{\mathbf{v}}[u_i(\mathbf{B}(\mathbf{v}))] \\
\geq &\sum_{i=1}^{n} \mathbb{E}_{v_i, \mathbf{v}'_{-i}} [u_i^{v_i}(a_i((v_i, \mathbf{v}'_{-i}), \mathbf{C}_{-i}), \mathbf{C}_{-i})] \\
= &\mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^{n} u_i^{v_i}(a_i(\mathbf{v}, \mathbf{C}_{-i}), \mathbf{C}_{-i}) \right] \text{(by relabeling } \mathbf{v}'_{-i} \text{ by } \mathbf{v}_{-i}) \\
\geq &\left(1 - \frac{1}{e}\right) \sum_{i=1}^{n} v_i(o_i^{\mathbf{v}}) - \sum_{j=1}^{m} \beta_j(\mathbf{C}) \\
= &\left(1 - \frac{1}{e}\right) \sum_{i=1}^{n} v_i(o_i^{\mathbf{v}}) - \sum_{j=1}^{m} \mathbb{E}_{\mathbf{v}}[\beta_j(\mathbf{B}(\mathbf{v}))]
\end{aligned}$$

So, $\mathbb{E}_{\mathbf{v}}[SW(\mathbf{B}(\mathbf{v}))] = \sum_i \mathbb{E}_{\mathbf{v}}[u_i(\mathbf{B}(\mathbf{v}))] + \sum_j \mathbb{E}_{\mathbf{v}}[\beta_j(\mathbf{B}(\mathbf{v}))] \geq \left(1 - \frac{1}{e}\right) \mathbb{E}_{\mathbf{v}}[SW(\mathbf{o}^{\mathbf{v}})]$. □

7.3 Lower bounds

7.3.1 Submodular valuations

Theorem 7.4. *The price of anarchy for submodular discriminatory auctions is at least* 1.099.

Proof. We present an example for a discriminatory auction with submodular valuations and show that the PoA of mixed Nash equilibria is at least 1.099.

We design a game with two players and two identical items. Player 1 has valuation (v, v), i.e., her valuation is v if she gets one or two items; whereas player 2 has valuation $(1, 2)$, i.e., he is additive with value 1 for each item. We use the following distribution functions defined by Hassidim et al. [20]:

$$G(x) = \frac{x}{1-x}, \qquad x \in [0, 1/2]; \qquad\qquad F(y) = \frac{v - 1/2}{v - y}, \qquad y \in [0, 1/2].$$

Consider the following mixed strategy profile. Player 1 bids $(x, 0)$ and player 2 bids (y, y), where x and y are drawn from $G(x)$ and $F(y)$, respectively. Noting that player 2 bids 0 with probability $F(0) = 1 - 1/2v$, we need a tie-breaking rule for the case of bidding 0, in which player 2 always gets the item. We claim that this mixed strategy profile is a Nash equilibrium.

First we prove that playing $(x, 0)$ for player 1 is a best response for every $x \in [0, 1/2]$. Notice that (x, x') with $x' \leq x$, is dominated by $(x, 0)$, since if player 1 gets at least one item, she should pay at least x and getting both items doesn't add to her utility.

$$u_1(x, 0) = F(x) \cdot (v - x) = v - 1/2.$$

Clearly, bidding higher than 1/2 guarantees the item but the payment is higher.

Now we need to show that (y, y) is a best response for player 2, for every $y \in [0, 1/2]$. Consider any strategy (y, y') with $y, y' \in [0, 1/2]$ and $y \geq y'$.

$$\begin{aligned}
u_2(y, y') &= \mathbb{P}[x \leq y'](2 - y - y') + \mathbb{P}[x > y'](1 - y) \\
&= G(y')(2 - y - y') + (1 - G(y'))(1 - y) = G(y')(1 - y') + 1 - y = 1 + y' - y \leq 1,
\end{aligned}$$

and $u_2(y, y) = 1$ is maximum possible. Bidding strictly higher than $1/2$ for both items is not profitable, since then her utility is $2 - 2y < 1$.

Now we calculate the expected social welfare of this Nash equilibrium.

$$\begin{aligned}
\mathbb{E}[SW] &= \mathbb{P}[y \geq x]2 + \mathbb{P}[x > y](1 + v) \\
&= 2 - (1 - v)\mathbb{P}[x > y] \\
&= 2 - (1 - v) \int_0^{1/2} F(x) dG(x)
\end{aligned}$$

This expression is maximized for $v = 0.643$. For this value of v, $\mathbb{E}[SW] = 1.818$. Since $SW(\mathbf{O}) = 2$, we get PoA$= 1.099$.[12] $\square$

7.3.2 Subadditive valuations

We provide a tight lower bound of 2 for subadditive valuations in discriminatory auctions which is similar to the lower bound of Section 4, adjusted to discriminatory auctions.

Theorem 7.5. *For discriminatory auctions the price of anarchy in mixed Nash equilibria is at least* 2 *for bidders with subadditive valuations.*

Proof. Consider two players and m items with the following valuations: player 1 is a unit-demand player with valuation $v < 1$ if she gets at least one item; player 2 has valuation 1 for getting less than m but at least one items, and 2 if she gets all the items. Inspired by [20], we use the following distribution functions:

$$G(x) = \frac{(m-1)x}{1-x}, \qquad x \in [0, 1/m]; \qquad\qquad F(y) = \frac{v - 1/m}{v - y}, \qquad y \in [0, 1/m].$$

Player 1 bids $b_1 = (x, 0, ..., 0)$ and player 2 bids $b_2 = (y, ..., y)$. x and y are drawn from $G(x)$ and $F(y)$, respectively. In case of a tie, the item is always allocated to player 2.

Let $\mathbf{B} = (B_1, B_2)$ denote this mixed bidding profile. We are going to prove that $\mathbf{B}$ is a mixed Nash equilibrium for every $v > 1/m$.

If player 1 bids any x in the range $(0, 1/m]$ for one item, she gets the item with probability $F(x)$, since a tie occurs with zero probability. Her expected utility is $F(x)(v - x) = v - 1/m$. So, for every $x \in (0, 1/m]$ her utility is $v - 1/m$. If player 1 picks x according to $G(x)$, her utility is still $v - 1/m$, since she bids 0 with zero probability. Bidding something greater than $1/m$ results in a utility less than $v - 1/m$. Regarding player 1, it remains to show that her utility when bidding for only one item is at least as high as her utility when bidding for more items. Suppose player 1 bids $(x_1, ..., x_m)$, where $x_i \geq x_{i+1}$, for $1 \leq i \leq m - 1$. Player 1 doesn't get any item if and only if $y \geq x_1$. So, with probability $F(x_1)$, she gets at least one item and she pays at least x_1. Therefore, her expected utility is at most $F(x_1)(v - x_1) = v - 1/m$, but it would be strictly less if she had nonzero payments for other items with positive probability. This means that bidding only x_1 for one item and zero for the rest of them dominates the strategy $(x_1, ..., x_m)$.

If player 2 bids y for all items, where $y \in [0, 1/m]$, she gets m items with probability $G(y)$ and $m-1$ items with probability $1-G(y)$. Her expected utility is $G(y)(2-my)+(1-G(y))(1-(m-1)y) = G(y)(1 - y) + 1 - (m - 1)y = 1$. Bidding something greater than $1/m$ results in utility less than 1. Suppose now that player 2 bids $(y_1, ..., y_m)$, where $y_i \geq y_{i+1}$ for $1 \leq i \leq m - 1$. If $x \leq y_m$, player

[12] This bound can be improved to 1.109 by a similar construction with 3 items. For large m the bound goes to one, so we do not believe that this construction is tight. That is the reason why we present here the simplest version of 2 items.

2 gets all the items; otherwise she gets $m-1$ items and she pays her $m-1$ highest bids. So, her utility is

$$
\begin{aligned}
& G(y_m)\left(2-\sum_{i=1}^{m} y_i\right)+(1-G(y_m))\left(1-\sum_{i=1}^{m-1} y_i\right) \\
= \;& G(y_m)(1-y_m)+1-\sum_{i=1}^{m-1} y_i \\
= \;& my_m+1-\sum_{i=1}^{m} y_i \\
\leq \;& my_m+1-\sum_{i=1}^{m} y_m = 1.
\end{aligned}
$$

Overall, we proved that $\mathbf{B}$ is a mixed Nash equilibrium. It is easy to see that the social welfare in the optimum allocation is 2. In this Nash equilibrium, player 2 bids 0 with probability $1-\frac{1}{mv}$, so, with at least this probability, player 1 gets one item.

$$
SW(\mathbf{B}) \leq \left(1-\frac{1}{mv}\right)(v+1)+\frac{1}{mv}2 = 1+v+\frac{1}{mv}-\frac{1}{m}
$$

If we set $v = 1/\sqrt{m}$, then $SW(\mathbf{B}) \leq 1+\frac{2}{\sqrt{m}}-\frac{1}{m}$. So, $PoA \geq \frac{2}{1+\frac{2}{\sqrt{m}}-\frac{1}{m}}$ which, for large m, converges to 2. □

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A Mixed Nash equilibria with submodular valuations

Here, we construct a group of instances with submodular valuations, which have at least one mixed Nash equilibrium. These instances are a generalization of the lower bound of Section 3.2. We believe that this construction is interesting in its own right.

Consider an instance with $n+1$ players and n^d items, where d divides n. We will refer to the first n players as the *real* players and to the last one as the *dummy* player. Let $[n]$ denote the set of integers $\{1, \ldots, n\}$. We define the set of items as $M = [n]^d$, that is, they correspond to all the different vectors $w = (w_1, w_2, ..., w_d)$ with $w_i \in [n]$. Intuitively, they can be thought of as the nodes of a d dimensional grid, with coordinates in $[n]$ in each dimension.

We divide the *real* players into d groups of n/d players. Let $g(i)$ denote the group that player i belongs to. We associate each group with one of the dimensions (directions) of the grid. In particular, for any fixed player i, his valuation for a subset of items $S \subseteq M$ is the size (number of elements) in the $d-1$-dimensional projection of S in direction $g(i)$, times v. Formally,

$$v_i(S) = v|\{w_{-g(i)} \mid \exists w_{g(i)} \ s.t. \ (w_{g(i)}, w_{-g(i)}) \in S\}|.$$

It is straightforward to check that v_i has decreasing marginal valuations, and is therefore submodular. The valuation of the *dummy* player for any subset of items $S \subseteq M$ is $(v-1)|S|$.

Given these valuations, we describe a mixed Nash equilibrium $\mathbf{B} = (B_1, \ldots, B_n)$ having a PoA $\left(1 - \left(1 - \frac{1}{n}\right)^{n + \frac{n}{d} - 1}\right)^{-1}$ which is arbitrarily close to $\frac{1}{1 - e^{-1 - \frac{1}{d}}}$, for large enough n. The dummy player bids $v-1$ for every item, and receives the item if all of the real players bid at most $v-1$ for it. The utility of the dummy player is always 0. For real players the mixed strategy B_i is the following. Every player i picks a number $\ell \in [n]$ uniformly at random, and an x according to the distribution with CDF

$$G(x) = (n-1)\left(\frac{1}{(v-x)^{\frac{1}{n-1}}} - 1\right),$$

where $x \in \left[v-1, v - \left(\frac{n-1}{n}\right)^{n-1}\right]$. Subsequently, he bids x for every item $w = (\ell, w_{-g(i)})$, with $w_{g(i)} = \ell$ as $g(i)^{th}$ coordinate, and bids $v-1$ for the rest of the items. only in case of $v-1$ bids for an item, in which case the *dummy* player gets the item.

Let $F(x)$ denote the probability that any bidder i gets a fixed item j, given that he bids $b_i(j) = x > v-1$ for this item, and the bids in $\mathbf{b}_{-i}$ are drawn from $\mathbf{B}_{-i}$ (due to symmetry, this probability is the same for all items $w = (\ell, w_{-g(i)})$). For every other player k, the probability that he bids $v-1$ for item j is $(n-1)/n$, and the probability that j is in his selected slice but he bids lower than x is $G(x)/n$. Multiplying over the $n-1$ other players, we obtain

$$F(x) = \left(\frac{G(x)}{n} + \frac{n-1}{n}\right)^{n-1} = \frac{\left(1 - \frac{1}{n}\right)^{n-1}}{v-x}.$$

Notice that v_i is an additive valuation restricted to the slice of items that player i bids for in a particular b_i. Therefore the expected utility of i when he bids x in b_i is $F(x)(v-x)$ for one of these items, and comprising all items $\mathbb{E}[u_i(b_i)] = n^{d-1}F(x)(v-x) = n^{d-1}\left(1 - 1/n\right)^{n-1}$.

Next we show that for $v \geq \left(1 - \frac{1}{n}\right)^{-\frac{n}{d}+1}$, $\mathbf{B}$ is a Nash equilibrium. In particular, the bids b_i in the support of B_i maximize the expected utility of a fixed player i.

First, we fix an arbitrary $w_{-g(i)}$, and focus on the set of items $C := \{(\ell, w_{-g(i)}) \mid \ell \in [n]\}$, which we call a *column* for player i. Recall that i is interested in getting only one item within C, on the other hand his valuation is additive over items from different columns. Observe that restricted to a fixed column, submitting any bid $x \in \left[v-1, v-\left(\frac{n-1}{n}\right)^{n-1}\right]$ for one arbitrary item results in the constant expected utility of $\left(1-\frac{1}{n}\right)^{n-1}$, whereas a bid higher than $v-\left(\frac{n-1}{n}\right)^{n-1}$ guarantees the item but pays more so the utility becomes strictly less than $\left(1-\frac{1}{n}\right)^{n-1}$ for this column.

We introduce two functions, $F_1(x)$ and $F_2(x)$,

$$F_1(x) = \left(\frac{G(x)}{n} + \frac{n-1}{n}\right)^{\frac{d-1}{d}n} = \left(\frac{1-\frac{1}{n}}{(v-x)^{\frac{1}{n-1}}}\right)^{\frac{d-1}{d}n},$$

$$F_2(x) = \left(\frac{G(x)}{n} + \frac{n-1}{n}\right)^{\frac{n}{d}-1} = \left(\frac{1-\frac{1}{n}}{(v-x)^{\frac{1}{n-1}}}\right)^{\frac{n}{d}-1}.$$

$F_1(x)$ denotes the probability that any bidder i gets a fixed item j, while the rest of the players in $g(i)$ bid $v-1$, given that player i bids $b_i(j)=x$ for this item, and the rest bids in $\mathbf{b}_{-i}$ are drawn from $\mathbf{B}_{-i}$. Similarly, $F_2(x)$ denotes the same probability while the players of all the other groups, apart from $g(i)$, bid $v-1$. Notice that $F(x)=F_1(x)F_2(x)$. In any fixed $\mathbf{b}_{-i}$, every other player of group $k \neq g(i)$ submits the same bid for all items in C, because either the whole C is in the current slice of k, and he bids the same value x, or no item from the column is in the slice and he bids $v-1$. Therefore, $F_1(x)$ for items in C are *fully dependent* distributions, whereas $F_2(x)$ for items in C are *independent* distributions.

We first show that if bidder i bids more than $v-1$ for at least two items in C, bidding $v-1+\varepsilon$ for all of these items, for a significantly small $\varepsilon>0$, is a better strategy. Suppose that bidder i bids $\mathbf{x}$, which is $x_j > v-1$ for every item $j \in R \subseteq C$ and $v-1$ for the rest, where k is the cardinality of R. Reorder the items in R in a way that the bids are in non-increasing order. We will use mathematical induction over k. Let $u_i(\mathbf{x},k)$ be the utility of player i for bidding $\mathbf{x}$, when the number of bids strictly greater than $v-1$ is k.

$$\begin{aligned}
E[u_i(\mathbf{x},2)] &= F(x_2)F_2(x_1)(v-x_1-x_2) + F(x_2)(1-F_2(x_1))(v-x_2) \\
&\quad + F_2(x_1)(F_1(x_1)-F(x_2))(v-x_1) \\
&= F(x_2)(v-x_2) + F(x_1)(v-x_1) - F(x_2)F_2(x_1)v \\
&= 2\left(1-\frac{1}{n}\right)^{n-1} - F(x_2)F_2(x_1)v
\end{aligned}$$

$E[u_i(\mathbf{x},2)]$ is maximized when both x_1 and x_2 are minimized, so for $x_1=x_2=v-1+\varepsilon$. Let R_{-1} denote the set of all k items apart from the first one (for which bidder i bids the most). Assume that $E[u_i(\mathbf{x},k-1)]$ is maximized when $x_j = v-1+\varepsilon$, for all $j \in R_{-1}$. Moreover, let $\mathbb{P}(S \neq \emptyset | S \subseteq R_{-1})$ and $\mathbb{P}(S)$ be the probabilities that he gets at least one item from R_{-1} and gets S, respectively. It is easy to see that $\mathbb{P}(S \neq \emptyset | S \subseteq R_{-1}) = \sum_{\substack{S \subseteq R_{-1} \\ S \neq \emptyset}} \mathbb{P}(S)$.

$$\begin{aligned}
E[u_i(\mathbf{x},k)] &= F_2(x_1)(F_1(x_1) - \mathbb{P}(S \neq \emptyset | S \subseteq R_{-1}))(v-x_1) \\
&\quad + F_2(x_1)\textstyle\sum_{S \subseteq R_{-1}, S \neq \emptyset} \mathbb{P}(S)(v - \sum_{j \in S} x_j - x_1) \\
&\quad + (1-F_2(x_1))E[u_i(\mathbf{x}_{-1},k-1)] \\
&= F(x_1)(v-x_1) - F_2(x_1)\mathbb{P}(S \neq \emptyset | S \subseteq R_{-1})(v-x_1) \\
&\quad + F_2(x_1)E[u_i(\mathbf{x}_{-1},k-1)] - F_2(x_1)x_1 \textstyle\sum_{S \subseteq R_{-1}, S \neq \emptyset} \mathbb{P}(S) \\
&\quad + (1-F_2(x_1))E[u_i(\mathbf{x}_{-1},k-1)] \\
&= \left(1-\tfrac{1}{n}\right)^{n-1} + E[u_i(\mathbf{x}_{-1},k-1)] - F_2(x_1)\mathbb{P}(S \neq \emptyset | S \subseteq R_{-1})(v-x_1) \\
&\quad - F_2(x_1)x_1\mathbb{P}(S \neq \emptyset | S \subseteq R_{-1}) \\
&= \left(1-\tfrac{1}{n}\right)^{n-1} + E[u_i(\mathbf{x}_{-1},k-1)] - F_2(x_1)\mathbb{P}(S \neq \emptyset | S \subseteq R_{-1})v
\end{aligned}$$

$F_2(x_1)$ is minimized when $x_1 = v - 1 + \varepsilon$, for a significantly small $\varepsilon > 0$, $E[u_i(\mathbf{x}_{-1}, k-1)]$ is maximized when $x_j = v - 1 + \varepsilon$, for all $j \in R_{-1}$ and $\mathbb{P}(S \neq \emptyset | S \subseteq R_{-1})$ is minimized for the same values. So, $E[u_i(\mathbf{x}, k)]$ is maximized when $x_j = v - 1 + \varepsilon$, for all $j \in R$.

We next prove that for $v \geq \left(1 - \frac{1}{n}\right)^{-\frac{n}{d}+1}$, bidding $x > v - 1$ only for one item dominates the strategy of bidding x for more than one items.

Lemma A.1. *For any integer $k \geq 1$,*

$$\sum_{r=1}^{k} \binom{k}{r} x^r (1-x)^{k-r} = 1 - (1-x)^k \text{ and } \sum_{r=1}^{k} \binom{k}{r} x^r (1-x)^{k-r} r = kx.$$

Proof.

$$\sum_{r=1}^{k} \binom{k}{r} x^r (1-x)^{k-r} = \sum_{r=0}^{k} \binom{k}{r} x^r (1-x)^{k-r} - \binom{k}{0} x^0 (1-x)^{k-0}$$
$$= (x + 1 - x)^k - (1-x)^k = 1 - (1-x)^k$$

$$\begin{aligned}
\sum_{r=1}^{k} \binom{k}{r} x^r (1-x)^{k-r} r &= \sum_{r=1}^{k} \frac{k!}{r!(k-r)!} x^r (1-x)^{k-r} r \\
&= k \sum_{r=1}^{k} \binom{k-1}{r-1} x^r (1-x)^{k-r} \\
&= kx \sum_{r=0}^{k-1} \binom{k-1}{r} x^r (1-x)^{k-1-r} \\
&= kx(x+1-x)^{k-1} = kx
\end{aligned}$$

$\square$

By using Lemma A.1, the utility of bidding x for k items is,

$$\begin{aligned}
E[u_i(x,k)] &= F_1(x) \sum_{r=1}^{k} \binom{k}{r} F_2(x)^r (1 - F_2(x))^{k-r} (v - rx) \\
&= F_1(x) \left(\left(1 - (1 - F_2(x))^k \right) v - kF_2(x)x \right).
\end{aligned}$$

We are going to bound the value of v, so that the utility decreases as k increases. So, we would like the following to hold.

$$\begin{array}{cc}
E[u_i(x,k+1)] - E[u_i(x,k)] & \leq 0 \\
F_1(x)\left(\left(1 - (1 - F_2(x))^{k+1}\right) v - (k+1)F_2(x)x\right) & \\
-F_1(x)\left(\left(1 - (1 - F_2(x))^{k}\right) v - kF_2(x)x\right) & \leq 0 \\
(1 - F_2(x))^k v - (1 - F_2(x))^{k+1} v - F_2(x)x & \leq 0 \\
F_2(x)(1 - F_2(x))^k v - F_2(x)x & \leq 0 \\
(1 - F_2(x))^k v - x & \leq 0
\end{array}$$

The quantity $(1 - F_2(x))^k v - x$ is maximized when x is minimized. Therefore,

$$\begin{aligned}(1 - F_2(x))^k v - x &\leq (1 - F_2(v-1))^k v - v + 1 \\ &\leq (1 - F_2(v-1))v - v + 1 \\ &= 1 - F_2(v-1)v.\end{aligned}$$

Thus, $E[u_i(x, k+1)] - E[u_i(x,k)] \leq 0$, if $v \geq F_2^{-1}(v-1) = \left(1 - \frac{1}{n}\right)^{-\frac{n}{d}+1}$. Hence, for $v \geq \left(1 - \frac{1}{n}\right)^{-\frac{n}{d}+1}$, $\mathbf{B}$ is a Nash equilibrium.

It remains to calculate the expected social welfare of $\mathbf{B}$, and the optimal social welfare. We define a random variable w.r.t. the distribution $\mathbf{B}$. Let $Z_j = v$ if one of the real players $1, \ldots, n$ gets item j, and $Z_j = v - 1$ if the *dummy* player gets the item. The expected social welfare is

$$\begin{aligned}\mathop{\mathbb{E}}_{\mathbf{b} \sim \mathbf{B}}[SW(\mathbf{b})] &= \sum_j E[Z_j] \\ &= n^d((1 - Pr(\text{no real player bids for j}))v + Pr(\text{no real player bids for j})(v-1)) \\ &= n^d\left(v - \left(1 - \frac{1}{n}\right)^n\right).\end{aligned}$$

Finally, we show that the optimum social welfare is $n^d v$. An optimal allocation can be constructed as follows: For each item $(w_1, w_2, ..., w_d)$ compute $r = (\sum_{i=1}^n w_i \mod n)$. Allocate this item to the player r. Similar to the Section 3.2, each player is allocated n^{d-1} items.

Therefore, the price of anarchy is $\frac{v}{v - (1-\frac{1}{n})^n}$. For $v = \left(1 - \frac{1}{n}\right)^{-\frac{n}{d}+1}$, the price of anarchy becomes $\frac{(1-\frac{1}{n})^{-\frac{n}{d}+1}}{(1-\frac{1}{n})^{-\frac{n}{d}+1} - (1-\frac{1}{n})^n}$ which for large n it converges to $\frac{1}{1-e^{-1-\frac{1}{d}}}$.

It is easy to see that the case of $d = n$, is the special case of Section 3.2, for which the Price of Anarchy is $\frac{1}{(1-(1-\frac{1}{n})^n)}$, and for large n converges to $\frac{1}{(1-\frac{1}{e})} \approx 1.58$.

B A lower bound example for the single item Bayesian PoA.

Bayesian equilibria were known to be inefficient [22]. Here, for the sake of completeness, we present a lower bound example for the Bayesian price of anarchy, with two players and only one item.

Theorem B.1. *For single-item auctions the PoA in Bayesian Nash equilibria is at least 1.06.*

Proof. In the lower-bound instance we have two bidders and only one item. The valuation of bidder 1 is always 1. Let $l = 1 - 2/e$, and $r = 1 - 1/e$. The valuation of bidder 2 is distributed according to the cumulative distribution function H :

$$H = \begin{cases} \frac{1}{e(1-l)} = \frac{1}{2}, & x \in [0, l] \\ \frac{1}{e(1-\frac{x+l}{2})} = \frac{2}{e(1-x)+2}, & x \in [l, 1]. \end{cases}$$

Observe that $v_2 = 0$ with probability $1/2$, and v_2 is distributed over $[l, 1]$ otherwise. Consider the following bidding strategy $\mathbf{B} = (B_1, B_2)$: B_1 has a uniform distribution on $[l, r]$ with CDF $G(x) = \frac{x-l}{r-l} = ex - e + 2$ on $[l, r]$; whereas the distribution B_2 is determined by the distribution of v_2 :

$$b_2(v_2) = \begin{cases} 0, & v_2 = 0 \\ \frac{v_2 + l}{2}, & v_2 \in [l, 1]. \end{cases}$$

Let $F(x)$ denote the CDF of b_2. We can compute the distribution of b_2 as follows. For $x \in [0, l)$, we have $F(x) = \mathbb{P}[b_2 \leq x] = \frac{1}{2}$; for $x \in [l, r]$ we have $F(x) = \mathbb{P}[b_2 \leq x] = \mathbb{P}[\frac{v_2+l}{2} \leq x] = \mathbb{P}[v_2 \leq 2x - l] = \frac{1}{e(1-x)}$. Finally, $F(x) = 1$ for $x \geq r$. In summary, $b_2 = 0$ with probability $1/2$, and is distributed over $[l, r]$ otherwise. On the other hand, b_1 is uniformly distributed over $[l, r]$. We do not need to bother about tie-breaking, since there are no mass points in $[l, r]$.

We prove next, that $\mathbf{B}$ is a Bayesian Nash equilibrium. Consider first player 2. If $v_2 = 0$, his utility is clearly maximized. If $v_2 \in [l, 1]$, then $\mathbb{E}[u_2(b_2)] = G(b_2)(v_2 - b_2)$. By straightforward calculation we obtain that over $[l, r]$ the function $G(x)(v_2 - x) = \frac{(x-l)(v_2-x)}{r-l}$ is maximized in $b_2 = (v_2 + l)/2$, so $b_2(v_2)$ is best response for bidder 2.

Consider now player 1. Given that over $[l, r]$ the distribution of b_2 is $F(x) = \frac{1}{e(1-x)}$, every bid $b_1 \in [l, r]$ is best response for player 1, since his utility $F(b_1)(1 - b_1) = 1/e$ is constant. Now we are ready to compute the social welfare of this Nash equilibrium.

$$SW(\mathbf{B}) = Pr[v_2 \leq l] \cdot 1 + \int_l^1 (v_2 \cdot G(b_2(v_2)) + 1 - G(b_2(v_2))) \cdot h(v_2) dv_2 \leq 0.942$$

So the PoA is at least 1.06. □

C Anonymity assumption

Recall that we assume that $q_j^w(x)$ (or $q_j^l(x)$) is the same for all bidders. Without the anonymity of $q_j^w(\cdot)$ over buyers, we can show by the following example that the PoA is unbounded. Suppose there is a single item to be sold to two players with valuation $v_1 = 1$ and $v_2 = \epsilon$. The losing payment is 0 for both players but the winning payments are different such that $q^w(x) = x$ for bidder 1 and $\bar{q}^w(x) = \epsilon \cdot x$ for bidder 2. If there is a tie, then the item is allocated to player 2. Now consider the bidding strategy $b_1 = b_2 = 1$. It is easy to see that it forms a Nash Equilibrium and has $PoA = 1/\epsilon$.